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SPACE VEHICLE GUIDANCE — A BOUNDARY VALUE FORMULATION

by ROBERT W. HUNT AND ROBERT SILBER
Aero-Astrodynamics Laboratory

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ABSTRACT

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A mathematical formulation of the problem of guiding one stage of a space vehicle is given as a boundary value problem in differential equations. One approach to the solution of this problem is to generate the Taylor's series expansion (in several variables) about a known solution. The theoretical nature of such solutions is discussed, and a method for numerically computing them is presented. This method entails the numerical integration of an associated system of differential equations, and can be used to obtain the solution to any desired degree of accuracy for points in a region to be defined. An extension of the method to the problem of guiding several stages of a space vehicle is also given, employing fundamental composite function theory.

AUTHOR

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ASTRODYNAMICS AND GUIDANCE THEORY DIVISION
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SUMMARY

The problem of guiding a space vehicle in flight reduces ultimately to the determination of an explicit basis for making the steering decision at each instant of flight. This amounts to the determination of the appropriate vehicular thrust vector as a function of (possibly) time, current state and current performance. Of course, this steering function must be such that mission fulfillment results in an appropriately optimum sense.

In many cases, following the application of an optimization theory, such as the calculus of variations, the steering function can be defined as the solution to a certain boundary value problem associated with the differential equations of motion and of optimal control. Certain of the initial values (initial values meaning values at current time) are to be determined as functions of the other initial values (and these other initial values are precisely the arguments of the steering function) such that the resulting subsequent motion leads eventually and optimally to mission fulfillment. (Mission fulfillment and optimality imply conditions to be met at a later time, thus resulting in a boundary value problem.)

The explicit solution of such a boundary value problem is generally dependent on the availability of the solution of the associated system of differential equations. However, techniques do exist involving high speed digital computers which can be applied in lieu of this solution.

This paper, which consists essentially of a mathematical formulation of the problem of guiding a space vehicle, presents a particularly general approach to solving the boundary value problem, amounting to the expansion of the steering

function in Taylor's series (in several variables) about a known solution. Using imbedding theorems of differential equations and implicit function theory, it is pointed out that under certain conditions the steering function is (a well-defined) analytic function of its arguments, thus assuring convergence of the obtained series. The truncated result is, of course, the guidance polynomial.

The method entails the numerical integration of the Jacobi (linear) system associated with the considered system of differential equations. This integration is carried out for values corresponding to the previously mentioned one known solution of the boundary value problem. The integration is then extended to "Jacobi systems" of (arbitrarily) higher order.

Extension of the method to several stages is presented, employing fundamental composite function theory.

The following three aspects of the analysis merit special emphasis:

(1) The guidance problem is formulated as one of a class of boundary value problems.

(2) The exactness of the results is in no way impaired by linearization and/or modification of the involved equations. The order to which the determination can be carried out is in no way restricted by the analysis.

(3) The method is in no way dependent on particular aspects and/or properties of the considered equations, except insofar as certain assumptions regarding the analytic nature of the equations and other related assumptions are made to insure existence and analyticity of solutions.

This report is divided into three major sections. The first section is intended primarily for the uninitiated reader, and consists of discussions of a fundamental nature concerning various pertinent aspects of analysis and of guidance theory. It is not necessary to have read the first section in order to read the latter two, but the reader who encounters difficulty in finding motivation or concrete examples for the material of the latter two sections is referred to the first section.

The second section is concerned with the abstract formulation of a certain type of boundary value problem stemming from the guidance problem. After a concise statement of the problem, the existence and properties of the solution are treated.

The third section deals with the methodology of the proposed numerical technique. A good understanding of the method should point up the significance of the second section.

SECTION I: SOME INTRODUCTORY CONSIDERATIONS

A. GENERAL

The material of this section is for the most part very elementary and can be found in many texts. However, the topics which are discussed are discussed with a view to their application in the next two sections. For this reason, many aspects of elementary definition and theory are given an emphasis which might not be found in the usual texts, and many properties and examples are exhibited correspondingly more explicitly.

The treatment of the topics in this section is not intended to be rigorous; the rigorous aspects are treated in the next section.

B. ON SOLUTIONS OF DIFFERENTIAL EQUATIONS

Consider the simple system

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -y_1 \end{aligned} \tag{1.1}$$

in which the dot indicates differentiation with respect to time (t). We shall use the solutions of this system to illustrate definitions and results related to more general systems. In fact, we consider the system (s):

$$\dot{y}_1 = f_1(y_1,\ldots, y_n, t)$$

$$\dot{y}_2 = f_2(y_1,\ldots, y_n, t)$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (s)$$

$$\dot{y}_n = f_n(y_1,\ldots, y_n, t) .$$

Definition 1: A solution to (s) on a t-interval [a, b] is a set of n functions $\varphi_1(t),\ldots, \varphi_n(t)$ which are differentiable on [a, b] and such that for each $i = 1, 2,\ldots, n$, the identity

$$\dot{\varphi}_i(t) \equiv f_i(\varphi_1(t),\varphi_2(t),\ldots, \varphi_n(t), t) \qquad (1.2)$$

holds on [a,b]. Note that the right hand side of (1.2) is formed by composition of the functions $\varphi_1(t),\ldots, \varphi_n(t)$ with the function $f_i(y_1,\ldots, y_n, t)$.

As an example, the two functions $\varphi_1(t)=\sin t$ and $\varphi_2(t)=-\cos t$ constitute a solution to (1.1) on every t-interval. It is clear that the general solution of (1.1) is given by the two functions

$$\begin{aligned} \Phi_1(t,A,B) &= A \cos t + B \sin t \\ \Phi_2(t,A,B) &= - A \sin t + B \cos t, \end{aligned} \qquad (1.3)$$

that is, for every pair (A,B) of real numbers, the functions $\Phi_1(t,A,B)$ and $\Phi_2(t,A,B)$, considered as functions of t, constitute a solution of (1.1) on every t-interval. Also, each solution of (1.1) is represented by (1.3) for some appropriate pair (A,B).

The statement that each pair of values (A,B) yields a pair of functions of t solving (1.1) means that

$$\begin{aligned} \frac{\partial \Phi_1}{\partial t} &= \Phi_2 \\ \frac{\partial \Phi_2}{\partial t} &= -\Phi_1 \end{aligned} \qquad (1.4)$$

these being equalities in t. But the assertion can in fact be made much stronger. For note that

$$\frac{\partial}{\partial t}\Phi_1(t,A,B) = - A \sin t + B \cos t \equiv \Phi_2(t,A,B)$$

$$\frac{\partial}{\partial t}\Phi_2(t,A,B) = - A \cos t - B \sin t \equiv -\Phi_1(t,A,B) , \tag{1.5}$$

these being identities <u>in all three arguments</u>.*

Because of these considerations, it is of interest to formulate the following definition:

<u>Definition 2</u>: Let $\Phi_1(t, a_1,\ldots, a_k)$, $\Phi_2(t, a_1,\ldots, a_k),\ldots,$ $\Phi_n(t, a_1,\ldots, a_k)$ be partially differentiable with respect to t for $t \epsilon [a,b]$ and for $(a_1,\ldots, a_k) \epsilon Z$, a subset of k-space. Let G be the k+1 dimensional set of arguments $(t, a_1,\ldots, a_k)$ for which $t \epsilon [a,b]$ and $(a_1,\ldots, a_k) \epsilon Z$; i.e., $G=[a,b]\times Z$. Then we say that the set of functions $\Phi_1(t, a_1,\ldots, a_k),\ldots, \Phi_n(t, a_1,\ldots, a_k)$ constitutes a solution of (s) on G if for each $i=1,2,\ldots,n$ and each $(t, a_1,\ldots, a_k) \epsilon G$,

*Of course, the identity (1.5) does not necessarily follow from (1.4). That is, the fact that certain choices for (A,B) yield functions of t satisfying the differential equations in t does not in itself imply that the equations will hold identically in the constants as well. This matter is greatly clarified by the augmented terminology "identically on S." That is, to say that a certain equation holds identically on S means that equality is satisfied for each argument (or set of arguments) in S. The function-theoretic implications of such a statement then become dependent on the topological properties of S. If S has sufficient properties, one may infer from the identity on S that two functions are indeed the same function and so be able to equate all derivatives as well. This is the case in (1.4) since the set S consists of <u>all</u> values of t, A and B.

$$\frac{\partial}{\partial t}\Big(\Phi_i(t,a_1,\ldots,a_k)\Big) = \qquad (1.6)$$

$$f_i(\Phi_1(t,a_1,\ldots,a_k),\ldots,\Phi_n(t,a_1,\ldots,a_k),t) \ .$$

The further relations which equation (1.6) will imply between the functions on the left and right sides of (1.6) depend on two things: the analytic nature of the functions themselves and the topological nature of the set G. These considerations are implicit in the treatment in part two of this paper, and so will not be treated here. However, to make the reader aware of the sort of considerations which are pertinent to later analysis, a little will be said about the example (1.1).

The functions $\Phi_1(t,A,B)$ and $\Phi_2(t,A,B)$ defined by (1.3) correspond to the functions $\Phi_1(t,a_1,\ldots,a_k),\ldots,$ $\Phi_n(t,a_1,\ldots,a_k)$ of definition 2. Condition (1.6) of definition 2 is fulfilled by (1.5). As has been pointed out, the functions $\partial/\partial t[\Phi_1(t,A,B)]$ and $\Phi_2(t,A,B)$ are indeed the <u>same</u> function, considered as functions of three arguments. This is, in this case, a consequence of (1.6), the fact that the functions in (1.5) are <u>analytic</u> and the set on which they are analytic consists <u>of all</u> complex t, all complex A and all complex B. It is therefore possible to equate all derivatives of the two functions.

The reader should be aware that it is easy to construct functions $\Phi_1(t,a_1,a_2)$ and $\Phi_2(t,a_1,a_2)$ which satisfy condition (1.6) on some set S, but such that the functions $\partial/\partial t[\Phi_1(t,A,B)]$ and $\Phi_2(t, a_1,a_2)$ do not turn out to be identical. For example, let

$$\Phi_1(t,a_1,a_2) = a_1 \sin t + t \sin a_2$$

$$\Phi_2(t,a_1,a_2) = a_1 \cos t - t \sin a_2.$$

then the conditions

$$\frac{\partial \Phi_1}{\partial t}(t,a_1,a_2) = \Phi_2(t,a_1,a_2) \text{ and}$$

$$\frac{\partial \Phi_2}{\partial t}(t,a_1,a_2) = -\Phi_1(t,a_1,a_2)$$

are met as required by definition 2 on the set G, consisting of any closed t-interval for [a,b] and the set Z of all pairs (a_1, a_2) with a_1 unrestricted but a_2 a multiple of π.

The point of all of this, once again, is that the strong identity given by (1.5) is not an automatic consequence of (1.6).

We now consider the constants A and B in (1.3). These constants might have been taken as "initial values" in the following sense. A time τ is selected as the "initial" time and η_1 and η_2 are selected as the values of y_1 and y_2, respectively, at $t=\tau$. Then (1.3) becomes

$$
\begin{aligned}
Y_1(t,\tau,\eta_1,\eta_2) &= \eta_1 \cos (t-\tau) + \eta_2 \sin (t-\tau) \\
Y_2(t,\tau,\eta_1,\eta_2) &= -\eta_1 \sin (t-\tau) + \eta_2 \cos (t-\tau)
\end{aligned}
\tag{1.7}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big(Y_1(t,\tau,\eta_1,\eta_2)\Big) &\equiv Y_2(t,\tau,\eta_1,\eta_2) \\
\frac{\partial}{\partial t}\Big(Y_2(t,\tau,\eta_1,\eta_2)\Big) &\equiv -Y_1(t,\tau,\eta_1,\eta_2)
\end{aligned}
\tag{1.8}
$$

these identities holding in all four arguments.

But along with the identities (1.8), there are two others, which are a consequence of the definition of η_1, η_2 and τ. These are

$$Y_1(\tau,\tau,\eta_1,\eta_2) \equiv \eta_1$$
$$Y_2(\tau,\tau,\eta_1,\eta_2) \equiv \eta_2 \ , \tag{1.9}$$

in which identity holds in all three arguments. This means, for instance, that Y_1 reduces to η_1 at $t = \tau$, regardless of the values assigned to τ, η_1, and η_2. A similar statement is applicable to Y_2.

These statements about initial values have their analogue in the general case. We summarize them in the following rather lengthy definition.

Definition 3: Consider a set of functions $Y_1(t, \tau, \eta_1,\ldots, \eta_n)$, $\ldots$, $Y_n(t, \tau, \eta_1,\ldots, \eta_n)$. Let Z be a subset of n-space and let Z' consist of all points $(\tau,\eta_1,\ldots,\eta_n)$ such that $\tau \in [a,b]$ and $(\eta_1,\ldots,\eta_n) \in Z$, i.e., $Z' = [a,b] \times Z$. Let $G = [a,b] \times Z'$, the set of all $(t,\tau,\eta_1,\ldots,\eta_n)$ such that $t \in [a,b]$, $\tau \in [a,b]$ and $(\eta_1,\ldots,\eta_n) \in Z$. Suppose that the set $Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots, Y_n(t,\tau,\eta_1,\ldots,\eta_n)$ constitutes a solution of (s) on G in the sense of definition 2. Suppose further that for each $\tau \in [a,b]$ and each $(\eta_1,\eta_2,\ldots,\eta_n) \in Z$ and each $i=1,2,\ldots,n$,

$$Y_i(\tau,\tau,\eta_1,\ldots,\eta_n) = \eta_i \ . \tag{1.10}^*$$

Then we say that τ is the initial time for the solution set $Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n)$ and that the arguments $(\eta_1,\ldots,\eta_n)$ are the initial values of the solution. This terminology is applicable, of course, on Z'.

---

* cf. (1.9) .

The discussion immediately following definition 2 is again of interest for definition 3. Without belaboring the point, it is simply mentioned that if the functions Y_i and the set Z' possess sufficiently many properties, equation (1.10) may be differentiated on both sides unrestrictedly. This is of interest later on.

Questions concerning the existence and properties of solutions such as those described in definitions 2 and 3 are not discussed here. They will, however, be treated in Section II of this paper for the particular type of system (s) of interest.

Since the set $Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n)$ of definition 3 is a solution of (s) on G in the sense of definition 2, equation (1.6) is applicable. This yields

$$\frac{\partial}{\partial t} Y_i(t,\tau,\eta_1,\ldots,\eta_n) = f_i(Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n),t) \tag{1.11}$$

for $i=1,2,\ldots,n$, and for $(t,\tau,\eta_1,\ldots,\eta_n) \in G$.

This identity and its partial derivatives together with (1.10) and its partial derivatives are basic to the method outlined in Section III. Once again it is pointed out that differentiation of these identities must be justified, but this will be done in Section II.

C. THE JACOBI EQUATIONS

In this section the partial derivatives of the functions $Y_i(t,\tau,\eta_1,\ldots,\eta_n)$ with respect to the parameters $(\tau,\eta_1,\ldots,\eta_n)$ will be considered. It will be convenient to assume that all partial derivatives which come under consideration in the present discussion exist and are continuous. Under this assumption, certain systems of equations which these partial derivatives satisfy will be derived.

Consider the identity (1.11). We will now take partial derivatives on both sides of this identity, assuming that all involved processess for equating of derivatives can be justified. Differentiation of the right side of (1.11) requires the chain rule, since the right side is a composite function.

Suppose both sides of (1.11) are differentiated partially with respect to η_k. The left side yields

$$\frac{\partial}{\partial \eta_k}\Big(\frac{\partial Y_i}{\partial t}(t,\tau,\eta_1,\ldots,\eta_n)\Big),$$

which, under the assumptions of existence and continuity of of second partials, can be written

$$\frac{\partial}{\partial t}\Big(\frac{\partial Y_i}{\partial \eta_k}(t,\tau,\eta_1,\ldots,\eta_n)\Big) .$$

On the right side, application of the chain rule yields

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j} \frac{\partial Y_j}{\partial \eta_k} (t,\tau,\eta_1,\ldots,\eta_n) \quad .$$

For purposes of subsequent differentiations, it is important that the reader realize explicitly the substitutions (and the order thereof) called for by the chain rule. In differentiating the right side of (1.11), one begins with the original functions which were composed to yield that function being differentiated. These are

$$f_i(y_1,\ldots,y_n,t)$$

and $\qquad Y_j(t,\tau,\eta_1,\ldots,\eta_n), \quad j=1,2,\ldots,n$.

One then forms $\dfrac{\partial f_i}{\partial y_j}$, which, of course, is again a function of $(y_1,\ldots,y_n,t)$ (<u>not</u> of $(t,\tau,\eta_1,\ldots,\eta_n)$).

After the differentiation of f_i with respect to y_j, the same substitutions are made for the arguments of $\partial f_i/\partial y_j$ as were made originally for f_i itself. Thus, for the considered differentiation, one substitutes into

$$\frac{\partial f_i}{\partial y_j}\,(y_1,\ldots,y_n,t)$$

to obtain

$$\frac{\partial f_i}{\partial y_j}\,\Big(Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n),t\Big)\ .$$

Thus, the equation for the considered partial derivative, written in such a way as to exhibit all arguments, is

$$\frac{\partial}{\partial t}\left(\frac{\partial Y_i}{\partial \eta_k}(t,\tau,\eta_1,\ldots,\eta_n)\right) = \tag{1.12}$$

$$\sum_{j=1}^{n}\frac{\partial f_i}{\partial y_j}\Big(Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n),t\Big)$$

$$\cdot\ \frac{\partial Y_j}{\partial \eta_k}(t,\tau,\eta_1,\ldots,\eta_n)\ .$$

Clearly it will be necessary, in the further use of (1.12), to introduce a shorter notation. Equation (1.12) has been so written because it is not possible to determine similar equations for second and higher partials, again by application of the chain rule, without this explicit statement of the involved arguments. For example, should one wish to differentiate the right hand side of (1.12) with respect to, say, η_p, it is clear from inspection of (1.12) that the chain rule would be applied to the term

$\frac{\partial f_i}{\partial y_j}$. Indeed, using arguments similar to those given in the derivation of (1.12), one obtains

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 Y_i}{\partial \eta_p \partial \eta_k}(t,\tau,\eta_1,\ldots,\eta_n)\right) = \tag{1.13}$$

$$\sum_{\ell=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f_i}{\partial y_\ell \partial y_j}\Big(Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n),t\Big) \cdot$$

$$\cdot \frac{\partial Y_j}{\partial \eta_k}(t,\tau,\eta_1,\ldots,\eta_n)\frac{\partial Y_\ell}{\partial \eta_p}(t,\tau,\eta_1,\ldots,\eta_n) +$$

$$+\sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}\Big(Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n),\ t\Big) \cdot$$

$$\cdot \frac{\partial^2 Y_j}{\partial \eta_p \partial \eta_k}(t,\tau,\eta_1,\ldots,\eta_n) \quad .$$

Clearly, under our assumptions of justifiability, higher and higher order equations of this form might be obtained by successive application of the chain rule. This is of significance to the method outlined in the third section.

It is interesting to note that the derivation of equations (1.12) and (1.13) in no way depended on that fact that the parameters $\eta_1,\ldots,\eta_n$ were initial values (cf. (1.10)). All that was used was the fact that the functions $Y_i(t,\tau,\eta_1,\ldots,\eta_n)$ were solutions of (s) in all their arguments (cf. (1.11)). From this it follows that (1.12) and (1.13) are applicable whenever one or more of the differentiations are with respect to τ.

Equation (1.12) might be written in matrix form. If Y is an $n \times n$ matrix with $\frac{\partial Y_i}{\partial \eta_j}$ appearing in the i^{th} row and j^{th} column, and if F is an $n \times n$ matrix with $\frac{\partial f_i}{\partial y_j}(Y_1, \ldots, Y_n, t)$ appearing in the i^{th} row and j^{th} column, then equation (1.12) reads

$$\frac{\partial}{\partial t} Y = FY \quad . \tag{1.14}$$

If (1.14) is considered for a particular solution of (s), that is, if the values of $\tau, \eta_1, \ldots, \eta_n$ are fixed in (1.14) so that every term becomes a function of t and t only, (1.14) may be written

$$\dot{Y} = FY \quad , \tag{1.15}$$

which is sometimes called the Jacobi equation associated to (s) and the particular solution considered.

The initial value of the Y matrix in (1.15) is known if it is supposed that the parameters $\eta_1, \ldots, \eta_n$ do in fact correspond to values at time τ. For by examination of (1.10) one sees that*

$$\begin{aligned} \left. \frac{\partial Y_i}{\partial \eta_j}(t, \tau, \eta_1, \ldots, \eta_n) \right|_{t=\tau} &= \frac{\partial}{\partial \eta_j} Y_i(\tau, \tau, \eta_1, \ldots, \eta_n) \\ &= \frac{\partial}{\partial \eta_j}(\eta_i) = \delta_{ij} \end{aligned} \tag{1.16}$$

*Once again assuming justifiability.

where δ_{ij} is the Kronecker delta. In terms of the matrix Y, this means that Y is the solution of

$$\dot{Y} = FY \tag{1.15}$$

which at $t=\tau$ is the $n \times n$ identity matrix. Also, again referring to (1.10),

$$\left.\frac{\partial^2 Y_i}{\partial\eta_k \partial\eta_j}(t,\tau,\eta_1,\ldots,\eta_n)\right|_{t=\tau} = \frac{\partial^2}{\partial\eta_k\partial\eta_j} Y_i(\tau,\tau,\eta_1,\ldots,\eta_n)$$

$$= \frac{\partial^2}{\partial\eta_k\partial\eta_j}(\eta_i) = 0 \tag{1.17}$$

for every $1 \leq i,j,k \leq n$; and it is clear that the values of all higher partials at $t=\tau$ are zero as well.

Before going on to the next topic, one more aspect of the Jacobi (and higher order) equations will be treated. In the method discussed in Section III, the following situation arises. A particular solution of (s) is under consideration; there is given a set of values $\tau^*, \eta_1^*, \ldots, \eta_n^*$. (We use asterisks to denote a fixed, chosen set of values.) Beginning at $t=\tau^*$ and ending at some final time $t_f^* > \tau^*$, the system (s) is integrated numerically, using a high speed digital computer, to yield the numerical values of $Y_i(t,\tau^*,\eta_1^*,\ldots,\eta_n^*)$ for each $i=1,2,\ldots,n$ and each $\tau^* \leq t \leq t_f^*$. Of course, it may be assumed that any other values which can be derived numerically from these values are also known. Thus, in particular, the matrix F of (1.15) is a numerically known function of time. Since the initial value of Y is known to be simply the $n \times n$ identity, and since Y satisfies (1.15), this equation can be integrated numerically to yield the values of Y at each $\tau^* \leq t \leq t_f^*$. In like manner, (1.13) can be integrated to yield numerical values for all second partials for $\tau^* \leq t \leq t_f^*$, and indeed, it is clear that numerical values for all partials (assuming their existence and justification of the method) can be determined for $\tau^* \leq t \leq t_f^*$ and, of course, corresponding to the particular initial values $\eta_1^*, \ldots, \eta_n^*$.

To illustrate this, consider the system (1.1) (whose solution is given by (1.7)). In order to properly analogize the general situation, we must pretend that (1.7) is not available. (Since (1.7) in fact is available, we have the advantage of knowing a priori what numbers will result from numerical integration from a particular set of initial values.)

Assume that for the particular solution determined by the values $\tau^*=0$, $\eta_1^*=1$, $\eta_2^*=1$, (1.1) is numerically integrated up to $t_f^*=\pi/2$. According to (1.7), for each $0 \le t \le \pi/2$, the numerical values so determined will be

$$\begin{aligned} Y_1(t,0,1,1) &= \cos t + \sin t \\ Y_2(t,0,1,1) &= -\sin t + \cos t \end{aligned} \qquad (1.18)$$

To set up equation (1.15) for this case, it is necessary to determine the F matrix. By comparing (1.1) with (s), we see that

$f_1(y_1,y_2,t) = y_2$ and $f_2(y_1,y_2,t) = -y_1$. Thus,

$$\frac{\partial f_1}{\partial y_1} = 0, \quad \frac{\partial f_1}{\partial y_2} = 1, \quad \frac{\partial f_2}{\partial y_1} = -1, \quad \frac{\partial f_2}{\partial y_2} = 0$$

and so

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and (1.15) becomes

$$\dot{Y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y, \qquad (1.19)$$

which, when written out, yields

$$\frac{d}{dt}\left(\frac{\partial Y_1}{\partial \eta_1}\right) = \frac{\partial Y_2}{\partial \eta_1}$$

$$\frac{d}{dt}\left(\frac{\partial Y_1}{\partial \eta_2}\right) = \frac{\partial Y_2}{\partial \eta_2}$$

$$\frac{d}{dt}\left(\frac{\partial Y_2}{\partial \eta_1}\right) = - \frac{\partial Y_1}{\partial \eta_1}$$

$$\frac{d}{dt}\left(\frac{\partial Y_2}{\partial \eta_2}\right) = - \frac{\partial Y_1}{\partial \eta_2}$$

If we for the moment denote $\frac{\partial Y_1}{\partial \eta_1}$, by α_1, $\frac{\partial Y_2}{\partial \eta_1}$ by α_2, $\frac{\partial Y_1}{\partial \eta_2}$ by β_1, and $\frac{\partial Y_2}{\partial \eta_2}$ by β_2, we see that the first and third equations above are the system

$$\dot{\alpha}_1 = \alpha_2$$

$$\dot{\alpha}_2 = - \alpha_1$$

while the second and fourth are

$$\dot{\beta}_1 = \beta_2$$

$$\dot{\beta}_2 = - \beta_1 \quad .$$

Clearly each of these systems is the same as (1.1) itself, except that the involved quantities have different designations. Applying (1.7) to α_1 and α_2 and then to β_1 and β_2, and observing from (1.16) that at

$t=\tau^*=0$, $\alpha_1=1$, $\alpha_2=0$, $\beta_1=0$, $\beta_2=1$, we see that $\alpha_1=\cos t$, $\alpha_2=-\sin t$, $\beta_1=\sin t$, $\beta_2=\cos t$. What we have thus shown is the following: If (1.19) is integrated numerically using the initial value (t=0) of Y to be the 2×2 identity, then the numerical value of the matrix Y thus determined will be, for each $0\leq t\leq \pi/2$, given by

$$Y = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

This result can be readily verified by differentiation of (1.7) and subsequent substitution of $\tau^*=0$, $\eta_1^*=1$, $\eta_2^*=1$.

This same numerical procedure can be applied to higher partials (cf. (1.13) and (1.17)). The result of this is that for any known solution of (s), the partial derivatives of the solution to the initial value problem (cf. definition 3) with respect to the initial values are numerically determinable along that solution. Already the idea of a Taylor's expansion about the known solution suggests itself!

D. THE GUIDANCE PROBLEM GENERALLY

Any discussion of space vehicle guidance must involve four major aspects. These are:

1. The flight environment,
2. Vehicle performance characteristics,
3. Mission,
4. Optimization criteria.

Mathematically, the first item amounts to the totality of extra-vehicular accelerations experienced by the vehicle. Examples are gravitational effects of one or more bodies and atmospheric lift and drag. The second item includes those parameters pertaining to the vehicle and influencing its motion. Certainly this would include thrust and mass of the vehicle. For atmospheric flight, the shape would also affect the motion.

The third item, the mission, is assumed to determine a number of mathematical relationships among the position and velocity variables of the vehicle and possibly time, whose simultaneous satisfaction is both a necessary and sufficient condition for mission fulfillment. These mathematical relationships are referred to as the mission criteria.

The last item, optimization criteria, results from the existence of a multiplicity of flight paths leading to mission fulfillment. When confronted with such a choice, it is natural to seek out the best or optimum solution, after having decided upon criteria for defining the optimum choice.

The exact way in which this can be done and a derivation and presentation of the involved equations is too lengthy a matter to treat here. Instead, we shall briefly describe the nature of the optimization and the results thereof. A detailed treatment of this problem may be found in the references listed at the end of this report.

For the sake of definiteness, we assume we have given a space vehicle with fixed performance. Let F be the magnitude of thrust, assumed constant, m be the mass of the vehicle at time t, assumed to vary linearly with time,* and x, y and z be the position coordinates of the vehicle in some coordinate system. The direction of the thrust vector requires two angles for its specification; let these be χ_p and χ_y. The convention for measuring these angles is not important here.

We assume the environment of the vehicle to be a known function of time. If the vehicle is subject to the gravitational effects of several bodies which are in motion, then time will appear explicitly in the equations governing the motion of the vehicle. In atmospheric flight, the components of velocity $\dot{x}$, $\dot{y}$, $\dot{z}$ will also be expected to influence the motion.

Again for the sake of definiteness, let us assume vacuum flight so that (neglecting relativistic considerations) the forces acting on the vehicle can be determined knowing F, m, t, x, y, z, χ_p, and χ_y. By an application of Newton's second law and division by $m \neq 0$, we can obtain

*It should be pointed out that these and later assumptions can be lifted.

$$\ddot{x} = f(F,m,t,x,y,z,\chi_p,\chi_y)$$

$$\ddot{y} = g(F,m,t,x,y,z,\chi_p,\chi_y)$$

$$\ddot{z} = h(F,m,t,x,y,z,\chi_p,\chi_y) \quad .$$

Actually, under our assumptions, F and m needn't appear explicitly as arguments, since they are functions of time and could be replaced by such. However, as parameters, they have important physical significance (especially their initial values) and it is desirable to retain their identity throughout. We do this as follows: to the above system is added the two differential equations

$$\dot{m} = c$$

and $\dot{F} = 0$.

Also, the original set of 3 second order equations is reduced to a system of 6 first order equations by the addition of the equations

$$\dot{x} = u$$

$$\dot{y} = v$$

$$\dot{z} = w \quad .$$

Upon combining all of these, we obtain the first order system

$$
\begin{aligned}
\dot{x} &= u \\
\dot{y} &= v \\
\dot{z} &= w \\
\dot{u} &= f(F,m,t,x,y,z,\chi_p,\chi_y) \\
\dot{v} &= g(F,m,t,x,y,z,\chi_p,\chi_y) \\
\dot{w} &= h(F,m,t,x,y,z,\chi_p,\chi_y) \\
\dot{m} &= c \\
\dot{F} &= 0 \,.
\end{aligned}
\tag{1.20}
$$

System (1.20) is not of the form of system (s) because of the presence of the variables χ_p, χ_y for which there are no corresponding differential equations. Thus, χ_p and χ_y are control variables. Their specification as functions of time, together with the specification of initial conditions would determine a unique solution to (1.20), if the right hand sides of (1.20) have enough properties. In a given situation, there are two outstanding questions to be answered concerning the control variables χ_p and χ_y. First, do there exist functions $\chi_p(t)$ and $\chi_y(t)$ which when coupled with appropriate initial values, result in mission fulfillment? To rephrase this, do there exist functions $\chi_p(t)$ and $\chi_y(t)$ which, when coupled with appropriate initial values, yield a solution of (1.20) which at some later instant satisfies (simultaneously) the mission criteria? The second question is meaningful only if the answer to the first is affirmative. It is: Is the set of functions $\chi_p(t)$, $\chi_y(t)$ unique, and if not, what set will yield mission fulfillment optimally?

Clearly, such questions can be formulated for systems more general than (1.20). One assumes as given a system

$$\begin{aligned}
\dot{x}_1 &= \varphi_1(x_1,x_2,\ldots,x_m,u_1,\ldots,u_k,t)\\
\dot{x}_2 &= \varphi_2(x_1,x_2,\ldots,x_m,u_1,\ldots,u_k,t)\\
&\vdots\\
\dot{x}_m &= \varphi_m(x_1,x_2,\ldots,x_m,u_1,\ldots,u_k,t)
\end{aligned} \tag{T}$$

and inquires as to the existence of functions $u_1(t),\ldots,u_k(t)$ which will take a set of initial values $x_{10},\ldots,x_{mo}$ for $x_1,\ldots,x_m$ into another set at a later time, or more generally, will yield solutions for $x_1(t),\ldots,x_m(t)$ which, at some later time, satisfy certain conditions of the form

$$F\,(t,x_1,\ldots,x_m) = 0 \quad .$$

If the answer is yes, the motion is said to be <u>controllable</u> at the initial point $x_{10},\ldots,x_{mo}$. If the functions $u_1(t),\ldots,u_k(t)$ are not unique, it is reasonable to optimize their selection.

In summary, there are two basic questions: existence and uniqueness, or equivalently, controllability and optimality.

We return now to system (1.20). Suppose that at some instant τ of flight, the vehicle has coordinates $x_o, y_o, z_o, u_o, v_o, w_o, F_o$, and m_o. Suppose that the mission criteria are given by a set of equations $F_i(t,x,y,z,u,v,w)=0$, $i=1,2,\ldots,k\leq 6$. The assumption $k\leq 6$ is a consequence of later results and will be explained at the appropriate point of the discussion.

Under the assumption of the existence of solutions for $\chi_p(t)$ and $\chi_y(t)$ yielding solutions of (1.20) which also satisfy the mission criteria, and under the assumption that the optimization criteria can be expressed as the minimization of some one function of the end-conditions* possessing enough properties to make it amenable to available mathematical techniques, an optimization theory such as the calculus of variations can be applied to system (1.20). This, for example, is the case if it is desired to minimize the propellant consumption entailed in reaching the end-point.

Now, the significant aspect of the application of calculus of variations to system (1.20) is that the result is to reduce the system back to the form of (s). More generally, a system of the form of (T) will become of the form of (s) by the addition of more differential equations. The control variables might be replaced by new variables, and the number might even be increased; but the end result is a system of the form of (s).

Therefore, <u>the analysis in Sections II and III begins with the system (s).</u> The actual optimization is of no interest for our present purposes once (s) is given along with certain boundary conditions to be discussed presently.

Now let $n=m+k$ and consider the system (s):

$$\begin{aligned} \dot{y}_1 &= f_1(y_1,\ldots,y_n,t) \\ &\vdots \\ \dot{y}_n &= f_n(y_1,\ldots,y_n,t)\,. \end{aligned} \tag{s}$$

*The values of x,y,t,u,v,w,t, etc., occurring at the instant at which the mission criteria are all satisfied are referred to as the <u>end-conditions</u>.

In the context of the guidance problem, the variables $y_1,\ldots,y_n$ are separated into two types: the first type, consisting of those variables in (1.20) for which there are initially given differential equations and initial values, will be denoted by $y_1,\ldots,y_m$. The remaining variables, consisting either of the original control variables, or those introduced by the optimization, are denoted by $y_{m+1},\ldots y_{m+k}$.

Again, by analogy to the guidance problem, we assume that the initial values at $t=\tau$ of $y_1,\ldots,y_m$ are given; let these be $\eta_1,\ldots,\eta_m$.

Without bothering (presently) about questions of uniqueness and existence, let $Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n)$ be the solution to (s) in terms of initial values, according to definition 3. Then, in order to solve the guidance problem, we must (1) keep the values of $\eta_1,\ldots,\eta_m$ fixed in order to reflect the actual initial values in flight and (2) find values of the remaining initial conditions, $\eta_{m+1},\ldots\eta_{m+k}$ which will yield a solution of (s) satisfying at some later instant the mission criteria (and other conditions imposed by the optimization to be mentioned shortly).

Note that every solution of (s) which at $t=\tau$ satisfies $y_i=\eta_i$, $i=1,2,\ldots,m,$ is given by the k-parameter family

$$y_i = Y_i(t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k}) \qquad (1.21)$$

This means that there are available exactly $k+1$ variables, namely $t,\eta_{m+1},\ldots,\eta_{m+k}$, with which to satisfy the mission criteria. Clearly the number of mission criteria cannot exceed $k+1$. This accounts for the earlier assumption of $k\leq 6$ in discussing system (1.20).

We now mention the other conditions imposed by the optimization. If the number of mission criteria is less than $k+1$, the calculus of variations, by way of a necessary condition known as the transversality condition, furnishes additional relationships to be met at the end-point (i.e., at the same instant that the mission criteria are satisfied) in such a way that the total number of conditions is invariably $k+1$*. The single difference between the mission criteria and the transversality condition is that the arguments of the

---

* This statement clearly assumes that the involved functions possess all those properties which may be required by the optimization theory. The transversality condition determines the optimum end point among those which satisfy the mission criteria.

conditions imposed by transversality generally range through all the variables $(t, y_1, \ldots, y_n)$ while the mission criteria contain at most the arguments $(t, y_1, \ldots, y_m)$.

For convenience, we group together the mission criteria and conditions of transversality into the one set

$$F_j(t, y_1, \ldots, y_n) = 0, \; j=1,2,\ldots,k+1 \; . \tag{1.22}$$

The analysis of Sections II and III is concerned with the functions obtained by the substitution of (1.21) into (1.22) and subsequent solution for the appropriate t and $\eta_{m+1}, \ldots, \eta_{m+k}$ in terms of $\eta_1{}^*, \ldots, \eta_m{}^*$. This is, after all, the needed information. The solutions obtained in this manner would yield that solution of (s) which was optimum or which was at least determined through satisfaction of conditions necessary for optimality and satisfaction of mission criteria.

Physically, the functions

$$\eta_{m+1}(\eta_1, \ldots, \eta_m), \ldots, \eta_{m+k}(\eta_1, \ldots, \eta_m) \text{ and } t(\eta_1, \ldots, \eta_m)$$

obtained by the above method of solution, amount to specification of the values of the control variables (χ_p and χ_y) at τ as functions of values of position, velocity, force, mass at τ and τ itself. The function $t(\eta_1, \ldots, \eta_m)$ represents the end time, at which the mission criteria and transversality condition are satisfied. For the significance of these functions, the reader is referred to the first paragraph of the abstract and introduction.

For later reference, we indicate the actual system to be solved for $\eta_{m+1}, \ldots, \eta_{m+k}$ and t:

$$F_j\Big(t, Y_1(t, \tau, \eta_1, \ldots, \eta_m, \eta_{m+1}, \ldots, \eta_{m+k}) \ldots, Y_n(\text{same arguments})\Big) = 0 \tag{1.23}$$

for each $j = 1, 2, \ldots, k+1$.

Thus, the equations (1.23) actually define the steering function implicitly in terms of initial state and performance. Existence, uniqueness, and other properties of the implicitly defined functions are directly determined by the properties of the functions in (1.22) and (1.21). Elegantly enough, though the functions in (1.21) are not known, their properties can be inferred from those of the system (s). Thus, with no actual knowledge of the general solution of (s) (cf.(1.20)), very definite statements can be made concerning the existence and nature of the steering functions. All of these statements will follow from examination of equations (1.23), (1.22), and (s), along with information concerning one numerically known solution of (s) satisfying (1.23).

SECTION II. ON THE FORMULATION AND SOLUTION OF A CERTAIN BOUNDARY-VALUE PROBLEM STEMMING FROM GUIDANCE CONSIDERATIONS

A. DEFINITIONS AND ASSUMPTIONS

Let there be given a system of first order differential equations

$$
\begin{aligned}
\dot{y}_1 &= f_1(y_1,\dots,y_n,t)\\
\dot{y}_2 &= f_2(y_1,\dots,y_n,t)\\
&\vdots\\
\dot{y}_n &= f_n(y_1,\dots,y_n,t)\ .
\end{aligned}
\qquad (s)
$$

If $(z_1,\dots,z_n,w)$ is an $n+1$-tuple of complex arguments, a polycylinder containing the complex $n+1$ dimensional point $(z_1,\dots,z_n,w)$ shall mean a set of the form

$$\left\{(z_1',\dots,z_n',w')\,\middle\|\, |z_i' - z_i| < r_i \text{ for each } i=1,2,\dots,n,\ |w'-w|<r_{n+1}\right\}$$

and $r_1,\dots,r_{n+1}$ are all positive. A function $f_i(z_1,\dots,z_n,w)$ will be said to be analytic at $(z_1,\dots,z_n,w)$ if f_i is expressible by a power series

$$\sum c^{(i)}_{\nu_1\nu_2\dots\nu_{n+1}}(z_1'-z_1)^{\nu_1}(z_2'-z_2)^{\nu_2}\dots(z_n'-z_n)^{\nu_n}(w'-w)^{\nu_{n+1}}$$

which is convergent at least in some polycylinder containing $(z_1,\ldots,z_n,w)$.

Let U be the set of <u>real</u> n+1-tuples $(y_1,\ldots,y_n,t)$ at which each of the functions $f_i(y_1,\ldots,y_n,t)$, $i=1,2,\ldots,n$, is analytic. Note that even though U contains only real n+1-tuples, analyticity entails convergence of the power series representation in the complex polycylinder.

Let there be given as well a set of functions

$$F_j(t,y_1,\ldots,y_n), \quad j=1,2,\ldots,k+1 \qquad (2.1)$$

and let V denote the collection of real n+1-tuples at which each of the functions $F_j(t,y_1,\ldots,y_n)$ is analytic.

Let the functions $\varphi_i(t)$, $i=1,2,\ldots,n$, be real-valued for each real $\tau^* \leq t \leq t_f^*$. Suppose further that they constitute a solution of (s) on $[\tau^*, t_f^*]$; i.e., each $\varphi_i(t)$, $i=1,2,\ldots,n$ is a differentiable function of t on $[\tau^*, t_f^*]$ and

$$\dot{\varphi}_i(t) = f_i(\varphi_1(t),\ldots,\varphi_n(t),t) \qquad (2.2)$$

for each $t \in [\tau^*, t_f^*]$, and each $i=1,2,\ldots,n$.

The right-hand side of (2.2) is obtained by the <u>composition</u> of each $\varphi_i(t)$ and $f_i(y_1,\ldots,y_n,t)$. For later notational convenience, we set

$$\varphi_i(\tau^*) = \eta_i^*, \quad i=1,2,\ldots,n\ . \qquad (2.3)$$

The solution set of $\varphi_1(t),\ldots,\varphi_n(t)$ will be said to be a <u>proper, real, analytic, non-singular, controllable solution</u> of (s) on $[\tau^*, t_f^*]$ if all of the following conditions are met (we shall abbreviate the above statement to saying that $\varphi_1(t),\ldots,\varphi_n(t)$ form a P.R.A.N.C. solution of (s) on $[\tau^*, t_f^*]$):

(i) $\varphi_i(t)$ is real-valued on $[\tau^*, t_f^*]$ for each $i=1,2,\ldots,n$ and satisfies (2.2) there;

(ii) for each $t \in [\tau^*, t_f^*]$, the point $(\varphi_1(t),\ldots,\varphi_n(t),t) \in U$, and $(t_f^*,\varphi_1(t_f^*),\ldots,\varphi_n(t_f^*)) \in V$;

(iii) $F_j(t_f^*,\varphi_1(t_f^*),\ldots,\varphi_n(t_f^*)) = 0$ for each $j=1,2,\ldots,k+1$;

(iv) $F_j(t,\varphi_1(t),\ldots,\varphi_n(t)) \neq 0$ for all j simultaneously for $t \in [\tau^*, t_f^*)$;

(v) a certain Jacobian $J \neq 0$. J will be defined shortly.

A word about terminology. In the expression proper, real, analytic, non-singular, controllable, each term has its basis in one property required by the definition. Property (i) is the basis for the term real solution; property (ii) is the basis for the term analytic; property (iii) is the basis for the term controllable; property (iv) is the basis for the term proper; and property (v) is the basis for the term non-singular. In order to define the Jacobian J of property (v), it is necessary to state the following theorem.

Theorem 1: Let the functions $\varphi_i(t)$, $i=1,2,\ldots,n$ be real-valued and differentiable on $[\tau^*, t_f^*]$ and satisfy (2.2) there. Suppose that for each $t \in [\tau^*, t_f^*]$, $(\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t),t) \in U$. Let $\varphi_i(\tau^*) = \eta_i^*$, $i=1,2,\ldots,n$. Then there exists a unique set of functions

$$Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n)$$

with the following properties:

(i) There is a positive number ρ such that for each $\tau \in [\tau^*, t_f^*]$ and each complex n-tuple $(\eta_1,\ldots,\eta_n)$ such that $|\eta_1-\varphi_1(\tau)|+|\eta_2-\varphi_2(\tau)|+\ldots+|\eta_n-\varphi_n(\tau)| < \rho$, the functions $Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n)$ constitute a solution to (2.2) on $[\tau^*, t_f^*]$.

(ii) $Y_i(\tau,\tau,\eta_1,\ldots,\eta_n) = \eta_i$ for each $i=1,2,\ldots,n$.

(iii) for each $i=1,2,\ldots,n$, $Y_i(t,\tau,\eta_1,\ldots,\eta_n)$ is an analytic function on the complex n+2 dimensional domain given by $t \in [\tau^*, t_f^*]$, $\tau \in [\tau^*, t_f^*]$, $|\eta_1-\varphi_1(\tau)|+|\eta_2-\varphi_2(\tau)|+\ldots+|\eta_n-\varphi_n(\tau)|< \rho$.

(iv) for each $t \in [\tau^*, t_f^*]$,

$$Y_i(t,\tau^*,\eta_1^*,\ldots,\eta_n^*) = \varphi_i(t); \; i=1,2,\ldots,n.$$

From property (i) of this theorem and (2.2), we have

$$\frac{\partial Y_i}{\partial t}(t,\tau,\eta_1,\ldots,\eta_n)=f_i(Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n),t) \; . \tag{2.4}$$

Let now, in the functions $Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,$ $Y_n(t,\tau,\eta_1,\ldots,\eta_n)$ of Theorem 1, the "initial values" $\eta_1,\ldots,\eta_n$ be divided into two groups: $\eta_1,\ldots,\eta_m$, and $\eta_{m+1},\ldots,\eta_{m+k}$, where $m+k=n$. We shall refer to the values $\eta_1,\ldots,\eta_m$ as the state parameters at τ and shall refer to $\eta_{m+1},\ldots,\eta_{m+k}$ as the control parameters at τ.

We now form the composite functions

$$F_j(t,Y_1(t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k}),\ldots,$$

$$Y_n(t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k}))\text{for each}$$

$j=1,2,\ldots,k+1$, and denote these functions of

$t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k}$ by

$$F_j^*(t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k}); \quad j=1,2,\ldots,k+1.$$

Define the Jacobian J to be

$$J = \frac{\partial(F_1^*,F_2^*,\ldots,F_{k+1}^*)}{\partial(t,\eta_{m+1},\ldots,\eta_{m+k})} \; .$$

The value of J obtained by setting $t=t_f^*$, $\tau=\tau^*$, and $\eta_i=\eta_i^*$ for each i=1,2,...n is that value of $J\neq 0$ required by property (v) of the definition of a P.R.A.N.C. solution.

B. STATEMENT AND SOLUTION OF A BOUNDARY-VALUE PROBLEM

We now formulate the basic problem of interest in this paper. Given values for the state parameters, to determine, as functions of these values, the control parameters and final time such that the conditions

$$F_j(t,y_1,\ldots,y_n) = 0; \quad j=1,2,\ldots,k+1 \qquad \text{(cf. 2.1)}$$

are met at the final time on the resultant solution. The way in which this problem is solvable and the nature of its solution is characterized by the following fundamental theorem.

Theorem 2: Suppose that the set of functions $\varphi_i(t)$, $i=1,2,\ldots,n$, constitutes a proper, real, analytic, non-singular, controllable solution of (2.2) on the interval $[\tau^*, t_f^*]$. Consider the system of equations

$$F_j^*(t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k}) = 0; \quad j=1,2,\ldots,k+1 \qquad (2.5)$$

in which the functions F_j^* are those previously defined. These equations implicitly define a set of functions

$$\begin{aligned} \eta_{m+r} &= \beta_r(\tau,\eta_1,\ldots,\eta_m), \qquad r = 1,2,\ldots,k \\ t &= t_f(\tau,\eta_1,\ldots,\eta_m) \end{aligned} \qquad (2.6)$$

for values of $(\tau,\eta_1,\ldots,\eta_m)$ neighboring $(\tau^*,\eta_1^*,\ldots,\eta_m^*)$ such that

$$F_j^*\Big(t_f(\tau,\eta_1,\ldots,\eta_m)\tau,\eta_1,\ldots,\eta_m,\beta_1(\tau,\eta_1,\ldots,\eta_m),\ldots, \beta_k(\tau,\eta_1,\ldots,\eta_m)\Big) = 0 \qquad (2.7)$$

for each $j=1,2,\ldots,k+1$ and for <u>all</u> complex arguments $(\tau,\eta_1,\ldots,\eta_m)$ neighboring (i.e., sufficiently close to) $(\tau^*,\eta_1^*,\ldots,\eta_m^*)$. The functions (2.6) are <u>analytic</u> and unique neighboring $(\tau^*,\eta_1^*,\ldots,\eta_m^*)$.

<u>Definition</u>: The functions $\beta_r(\tau,\eta_1,\ldots,\eta_m)$ are called the <u>control laws</u> and $t_f(\tau,\eta_1,\ldots,\eta_m)$ is called the <u>final time</u> (or time of cutoff).

<u>Remarks</u>: All essential theory is contained in Theorem 2. This theorem tells us that near a P.R.A.N.C. solution, the guidance problem is uniquely solvable and that the control laws are indeed analytic functions of the state parameters. This being the case, these laws have unique, convergent Taylor's series in their arguments, whenever these arguments neighbor those on the P.R.A.N.C. solution.

The next section is devoted to a numerical method for calculating this series. Henceforth, it is assumed that there has been determined <u>numerically</u> a P.R.A.N.C. solution of (2.2), and this solution will be denoted, as in this section, by $\varphi_i(t)$ for $i=1,2,\ldots,n$ and $t \in [\tau^*, t_f^*]$. All functions and equations of this section will be used in their already established context. However, for brevity, the numerically known P.R.A.N.C. solution will be referred to simply as a "reference trajectory."

SECTION III:
THE NUMERICAL GENERATION OF THE TAYLOR'S SERIES FOR THE CONTROL LAWS NEIGHBORING A REFERENCE TRAJECTORY

A. <u>INTRODUCTION</u>:

Given that one proper, real, analytic, non-singular, controllable solution of (2.2) is known numerically*, and this solution is henceforth referred to as the reference trajectory and denoted by the set $\varphi_1(t),\ldots,\varphi_n(t)$ on $[\tau^*, t_f^*]$, the control laws $\beta_r(\eta_1,\ldots,\eta_m)$, $r=1,2,\ldots,k$

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*The numerical determination of such a solution on a digital computer could be accomplished, for example, by the numerical integration of (s) from some set of initial values and subsequent iterations on the initial values until the conditions $F_j(t,y_1,\ldots,y_n) = 0$, $j=1,2,\ldots,k+1$ are simultaneously satisfied.

and the final time $t_f(\eta_1,\ldots,\eta_m)$ can be expanded in a Taylor's series in several variables about the "known" solutions $\beta_r(\tau^*,\eta_1^*,\ldots,\eta_m^*) = \eta_{m+r}^*$, $r=1,2,\ldots,k$, $t_f(\tau^*,\eta_1^*,\ldots,\eta_m^*) = t_f^*$. The series is obtained by the determination of the partial derivatives of the functions $\beta_r(\tau,\eta_1,\ldots,\eta_m)$, $r=1,2,\ldots,k$ and $t_f(\tau,\eta_1,\ldots,\eta_m)$ with respect to their arguments, at $(\tau^*,\eta_1^*,\ldots,\eta_m^*)$. The values of these partials are determined even though the functions themselves are defined only implicitly. From Section II it is known that all partials exist and that the Taylor's series is convergent, for one of the conclusions of Theorem 2 is, in fact, that the control laws and the final time are analytic at and near $(\tau^*,\eta_1^*,\ldots,\eta_m^*)$.

Three fundamental identities will form the basis for the entire procedure. Clearly, one of these must be the implicitly defining relations for the functions to be expanded. These are given by (2.7), repeated here for convenience.

$$F_j^*\Big(t_f(\tau,\eta_1,\ldots,\eta_m),\tau,\eta_1,\ldots,\eta_m,\beta_1(\tau,\eta_1,\ldots,\eta_m),\ldots,\beta_k(\tau,\eta_1,\ldots,\eta_m)\Big) = 0 \tag{3.1}$$

for each $j=1,2,\ldots,k+1$.

Recall that the functions F_j^* were defined by

$$F_j^*(t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k}) = F_j\Big(t,Y_1(t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k}),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k})\Big) \tag{3.2}$$

for each $j=1,2,\ldots,k+1$.

Recall that the functions $Y_i(t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k})$, $i=1,2,\ldots,n$ are defined by Theorem 1 of Section II and are uniquely defined and analytic functions for arguments

$(t,\tau,\eta_1,\ldots,\eta_n)$ such that $t \in [\tau^*, t_f^*]$, $\tau \in [\tau^*, t_f^*]$ and $|\eta_1-\varphi_1(\tau)|+|\eta_2-\varphi_2(\tau)|+ \ldots +|\eta_n-\varphi_n(\tau)|<\rho$. It is clear that as functions of t, these solutions of (2.2) can be continued to some values of $t > t_f^*$, because of analyticity at t_f^*.

The second fundamental identity is that one which characterizes the parameters $(\tau,\eta_1,\ldots,\eta_n)$ as being "initial values." This is given in Theorem 1 of the previous section, and is repeated here for convenience.

$$Y_i(\tau,\tau,\eta_1,\ldots,\eta_n) = \eta_i, \quad i=1,2,\ldots,n. \tag{3.3}$$

Again, this identity can be differentiated throughout an unlimited number of times because the left-hand side, being given by the composition of the analytic functions $Y_i(t,\tau,\eta_1,\ldots,\eta_n)$ with the analytic function $t=\tau$, is again analytic near $(\tau^*,\tau^*,\eta_1^*,\ldots,\eta_n^*)$. That the derivatives of the two sides of (3.3) can be equated is a consequence of having equality of the analytic functions $Y_i(\tau,\tau,\eta_1,\ldots,\eta_n)$ and η_i not just at $(\tau^*,\tau^*,\eta_1^*,\ldots,\eta_n^*)$, but for all $\tau \in [\tau^*, t_f^*]$ and all complex $(\eta_1,\ldots,\eta_n)$ satisfying $|\eta_1-\varphi_1(\tau)|+ \ldots +|\eta_n-\varphi_n(\tau)| < \rho$.

The third identity is that which the functions $Y_i(t,\tau,\eta_1,\ldots,\eta_n)$ must satisfy in order to be solutions of (2.2). This identity, given here for convenience, is stated following Theorem 1 of the previous section by (2.4).

$$\frac{\partial Y_i}{\partial t}(t,\tau,\eta_1,\ldots,\eta_n) = f_i(Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Y_n(t,\tau,\eta_1,\ldots,\eta_n),t) \tag{3.4}$$

for each $i=1,2,\ldots,n$.

Summarizing, each of the three identities (3.1), (3.3), and (3.4) consist of analytic functions neighboring the conditions $\tau=\tau^*$, $\eta_i=\eta_i^*$, $i=1,2,\ldots,n$, and $t \epsilon [\tau^*, t_f^*]$. Other identities are obtainable from these by unlimited differentiations.

B. DESCRIPTION OF THE METHOD

Recall that the partial derivatives which are needed to evaluate the coefficients for the first order terms of the desired Taylor's series of the control laws and final time are

$$\frac{\partial \beta_r}{\partial \eta_j} (\tau^*, \eta_1^*, \ldots, \eta_m^*)$$

for $r = 1, 2, \ldots, k$; $j = 1, 2, \ldots, m$,

$$\frac{\partial \beta_r}{\partial \tau} (\tau^*, \eta_1^*, \ldots, \eta_m^*) \quad \text{for } r=1,2,\ldots,k \quad \text{and}$$

$$\frac{\partial t_f}{\partial \eta_j} (\tau^*, \eta_1^*, \ldots, \eta_m^*) \quad \text{for } j=1,2,\ldots,m \quad \text{and}$$

$$\frac{\partial t_f}{\partial \tau} (\tau^*, \eta_1^*, \ldots, \eta_m^*) \ .$$

Bearing this in mind, differentiate both sides of (3.1) with respect to η_ℓ for arbitrary but fixed $1 \leq \ell \leq m$. This requires the chain rule and equation (3.2). The result is

$$\frac{\partial F_j}{\partial t} \frac{\partial t_f}{\partial \eta_\ell} + \sum_{p=1}^{n} \frac{\partial F_j}{\partial y_p} \frac{\partial Y_p}{\partial \eta_\ell} = 0, \text{ in which}$$

$$\frac{\partial Y_p}{\partial \eta_\ell} = \frac{\partial Y_p}{\partial t}\frac{\partial t_f}{\partial \eta_\ell} + \frac{\partial Y_p}{\partial \eta_\ell} + \sum_{r=1}^{k} \frac{\partial Y_p}{\partial \eta_{m+r}}\frac{\partial \beta_r}{\partial \eta_\ell} .$$

Combining these, we obtain

$$\frac{\partial F_j}{\partial t}\frac{\partial t_f}{\partial \eta_\ell} + \sum_{p=1}^{n} \frac{\partial F_j}{\partial y_p}\left[\frac{\partial Y_p}{\partial t}\frac{\partial t_f}{\partial \eta_\ell} + \frac{\partial Y_p}{\partial \eta_\ell} + \sum_{r=1}^{k} \frac{\partial Y_p}{\partial \eta_{m+r}}\frac{\partial \beta_r}{\partial \eta_\ell}\right] = 0 . \tag{3.5}$$

Several comments need to be made concerning the above identity. First of all, for purposes of subsequent differentiations, it is essential to note carefully the arguments associated with each of the partials appearing in (3.5). The reader who is in doubt concerning this matter is referred to Part C of Section I.

Secondly, note that (3.5) is valid for each

$1 \leq \ell \leq m$ and each $1 \leq j \leq k+1$.

An equation similar to (3.5) is obtained by differentiation of (3.1) with respect to τ. This yields

$$\frac{\partial F_j}{\partial t}\frac{\partial t_f}{\partial \tau} + \sum_{p=1}^{n} \frac{\partial F_j}{\partial y_p}\left[\frac{\partial Y_p}{\partial t}\frac{\partial t_f}{\partial \tau} + \frac{\partial Y_p}{\partial \tau} + \sum_{r=1}^{k} \frac{\partial Y_p}{\partial \eta_{m+r}}\frac{\partial \beta_r}{\partial \tau}\right] = 0 , \tag{3.6}$$

which is valid for each $1 \leq j \leq k+1$.

Let now, in (3.5) and (3.6), the values $t=t_f^*$, $\tau=\tau^*$, $\eta_i=\eta_i^*$ for $i=1,2,\ldots,m$ be substituted for the arguments of the involved partials. Then, and this is the kernel of the entire formulation, equations (3.5) and (3.6) will yield, in a manner to be described, linear systems which determine precisely the numerical values of those first order partial derivatives necessary for the determination of the coefficients in the Taylor's expansion of the control laws, as described in Part A of Section III. To see how this is so, consider equation (3.5) for $\ell=1$. As j runs from 1 to k+1, there results a set of k+1 equations involving the desired unknown partial derivatives

$$\frac{\partial t_f}{\partial \eta_1}(\tau^*,\eta_1^*,\ldots,\eta_m^*),\ \frac{\partial \beta_1}{\partial \eta_1}(\tau^*,\eta_1^*,\ldots,\eta_m^*),\ldots,\frac{\partial \beta_k}{\partial \eta_1}(\tau^*,\eta_1^*,\ldots,\eta_m^*).$$

This constitutes a system of k+l linear equations in k+l unknowns. The determination of the solution of this system is contingent upon the availability of numerical values for all other quantities appearing in the equations and upon the nonvanishing of the Jacobian, J, of the system. Satisfaction of the latter condition is implicit in the definition of a reference trajectory, but it is in fact the case that the other numerical values of the former condition are not at this point available. The remaining parts of Section III will be devoted to the description of a technique for obtaining these values.

Similar statements are applicable to (3.5) as ℓ assumes each of the remaining values 2, 3,...,m. Also, (3.6) yields a system of k+l equations linear in the k+l unknowns

$$\frac{\partial t_f}{\partial \tau},\ \frac{\partial \beta_r}{\partial \tau},\ r = 1,\ 2,\ldots,k.$$

Summarizing, we have seen that differentiation of (3.1) yields a non-singular linear system whose solutions furnish "first order" numerical values desired for the generation of the Taylor's series for the control laws and final time. The solution can be effected once the numerical values of all other quantities appearing in the system are known.

Before going to Part C and the determination of the numerical values of the other quantities appearing in the various linear systems, consider the systems obtained by subsequent differentiations of (3.1). These clearly involve second and higher order partial derivatives of the control laws and final time, both pure and mixed. Upon performing one such (arbitrary) differentiation, it becomes more or less apparent that the totality of possible differentiations of (3.1) with respect to the involved arguments yields non-singular linear systems, the solutions of which yield, assuming the availability (once again) of numerical values for all other partials appearing in the systems, all numerical values of the partial derivatives necessary for the generation of the Taylor's series for the control laws and final time.

The equations resulting from higher order differentiations of (3.5) and (3.6) are rather lengthy, but nevertheless straightforward. For example, suppose it is desired to differentiate (3.5) with respect to η_s. The very first factor appearing in (3.5) is

$$\frac{\partial F_j}{\partial t},$$

which, according to the chain rule, has for its arguments exactly those of the left side of (3.1). This means that differentiation of

$$\frac{\partial F_j}{\partial t}$$

results in an expression of the length of that of (3.5) itself. In fact,

$$\frac{\partial}{\partial \eta_s}\left(\frac{\partial F_j}{\partial t}\right) = \frac{\partial^2 F_j}{\partial t^2}\frac{\partial t_f}{\partial \eta_s} + \sum_{p=1}^{n} \frac{\partial^2 F_j}{\partial t \partial y_p}\left[\frac{\partial Y_p}{\partial t}\frac{\partial t_f}{\partial \eta_s} + \frac{\partial Y_p}{\partial \eta_s} + \sum_{r=1}^{k} \frac{\partial Y_p}{\partial \eta_{m+r}}\frac{\partial \beta_r}{\partial \eta_s}\right].$$

Similar differentiations are involved for the other terms of (3.5). But the result is again a linear system, this time in the unknown second partials. The Jacobian of this system is, as in every case, the same Jacobian $J \neq 0$ defined in Section II in connection with non-singularity of the reference trajectory. It is clear that certain other second order partials must be known numerically before solution for the desired partials can be effected, just as in the case of the first partials. And as more and more differentiations of (3.1) are carried out, the solution of the linear systems for higher and higher order desired partials will require prior determination of numerical values for other partials appearing in the identities of increasingly higher order.

We now turn to this determination.

C. THE NUMERICAL DETERMINATION OF PARTIAL DERIVATIVES NEEDED FOR THE SOLUTION OF THE LINEAR SYSTEMS OF B

Let us first decide what numerical values are needed in order to set up any of the linear systems in B. Since every linear system results from differentiation of (3.1), which in turn is dependent upon (3.2) for its definition, examination of these two identities should yield the desired information.

For purposes of the present discussion and for the sake of brevity, let all of the arguments appearing in (3.1) (these are $\tau,\eta_1,\ldots,\eta_m$) be denoted by the vector α, where $\alpha = (\alpha_0,\alpha_1,\ldots,\alpha_m)$ with $\alpha_0 = \tau$, $\alpha_i = \eta_i$, $i=1,2,\ldots,m$. Let the functions $\beta_1(\alpha)$, $\beta_2(\alpha),\ldots,\beta_k(\alpha)$ (the control laws) be combined into the single vector function $\beta(\alpha)$ with component functions $(\beta_1(\alpha),\ldots,\beta_k(\alpha))$. Finally, combine the functions $Y_1,\ldots,Y_n$ into the single n-dimensional vector Y. With this new notation, combine (3.1) and (3.2) to obtain

$$F_j\Big(t_f(\alpha),Y\big(t_f(\alpha),\alpha,\beta(\alpha)\big)\Big) = 0, \quad j=1,2,\ldots,k+1 . \tag{3.7}$$

In this new notation, (3.5) and (3.6) differ only in the index of the argument with respect to which differentiation is being carried out. For convenience, we rewrite them in the new notation.

$$\frac{\partial F_j}{\partial t}\frac{\partial t_f}{\partial \alpha_\ell} + \sum_{p=1}^{n}\frac{\partial F_j}{\partial y_p}\left[\frac{\partial Y_p}{\partial t}\frac{\partial t_f}{\partial \alpha_\ell} + \frac{\partial Y_p}{\partial \alpha_\ell} + \sum_{r=1}^{k}\frac{\partial Y_p}{\partial \eta_{m+r}}\frac{\partial \beta_r}{\partial \alpha_\ell}\right] = 0 \tag{3.8}$$

for each $j=1,2,\ldots,k+1$.

Since this system is to be solved for the terms $\frac{\partial t_f}{\partial \alpha_\ell}$, $\frac{\partial \beta_r}{\partial \alpha_\ell}$, $r=1,2,\ldots,k$, the quantities to be determined are

$$\frac{\partial F_j}{\partial t}, \frac{\partial F_j}{\partial y_p}, \frac{\partial Y_p}{\partial t}, \frac{\partial Y_p}{\partial \alpha_\ell}, \text{and } \frac{\partial Y_p}{\partial \eta_{m+r}} .$$

Differentiation of (3.8) with respect to α_s will yield a linear system to be solved for $\frac{\partial^2 t_f}{\partial \alpha_s \partial \alpha_\ell}$, $\frac{\partial^2 \beta_r}{\partial \alpha_s \partial \alpha_\ell}$, $r=1,2,\ldots,k+1$.

Without writing out the entire formula, we observe that upon application of the chain rule, there results an identity involving

$\frac{\partial^2 F_j}{\partial t^2}$, $\frac{\partial^2 F_j}{\partial y_p \partial t}$, $\frac{\partial^2 F_j}{\partial y_p \partial y_q}$, $\frac{\partial^2 Y_p}{\partial t^2}$, $\frac{\partial^2 Y_p}{\partial t \partial \alpha_s}$, $\frac{\partial^2 Y_p}{\partial t \partial \eta_{m+r}}$, $\frac{\partial^2 Y_p}{\partial \alpha_\ell \partial \alpha_s}$,

$\frac{\partial^2 Y_p}{\partial \alpha_\ell \partial \eta_{m+r}}$, $\frac{\partial^2 Y_p}{\partial \eta_{m+r} \partial \eta_{m+u}}$, along with first partials which can be assumed known from the solution of (3.8).

It is clear that this list includes every possible second partial of F_j with respect to combinations of the arguments $(t, y_1, \ldots, y_n)$ and every possible second partial of the Y_p with respect to combinations of the arguments $(t, \alpha_o, \ldots, \alpha_m, \eta_{m+1}, \ldots, \eta_{m+k})$. A similar statement holds for all higher order systems; it is in every case necessary to obtain numerical values for all partial derivatives of the F_j and Y_p of a given order.

The easiest and most directly obtainable of these are the partials of the F_j. For it is assumed that the equations $F_j(t, y_1, \ldots, y_n) = 0$, $j=1,2,\ldots,k+1$ are explicitly known (in the guidance problem either as given mission criteria or as derived transversality conditions). Therefore, these expressions can be differentiated to obtain explicit expressions for their partials. These are then evaluated for $t=t_f^*$, $\alpha_o=\tau^*$, $\alpha_i=\eta_i^*$, $i=1,2,\ldots,m$, $\eta_{m+r}=\eta_{m+r}^*$, $r=1,2,\ldots,k$. This amounts to insertion of the values of $y_1, \ldots, y_n$ at the end point of the reference trajectory, which is numerically known. In this way, all partials of the F_j can be obtained numerically.

The determination of the partials of the Y_p is more involved because the functions Y_p are not available. The description of the method for determining the partials will consist of a complete and detailed treatment of all involved first and second order partials, followed by appropriate comments concerning the general case. The general order and type of partial is not treated in detail for two reasons: first, the generalization of the procedure is relatively straightforward, and secondly, the notation and length of formulas for higher order cases are somewhat problematic.

The functions Y_p, by virtue of being solutions of (s), must satisfy (3.4). Reverting to the original notation, let (3.4) be differentiated throughout with respect to η_j. Then subsequent substitution of the values $\tau = \tau^*$, $\eta_1=\eta_1^*,\ldots,\eta_n=\eta_n^*$ results in an equation involving only time. This equation is, in fact,

$$\frac{d}{dt}\left(\frac{\partial Y_i}{\partial \eta_j}\right) = \sum_{u=1}^{n} \frac{\partial f_i}{\partial y_u}\frac{\partial Y_u}{\partial \eta_j}, \quad i,j=1,2,\ldots,n \,. \tag{3.9}$$

A similar equation results from differentiation with respect to τ:

$$\frac{d}{dt}\left(\frac{\partial Y_i}{\partial \tau}\right) = \sum_{u=1}^{n} \frac{\partial f_i}{\partial y_u}\frac{\partial Y_u}{\partial \tau}, \quad i=1,2,\ldots,n. \tag{3.10}$$

Equation (3.9) for every $i,j=1,2,\ldots,n$ and equation (3.10) for every $i=1,2,\ldots,n$ can be combined into a single matrix equation as follows. Let F be the $n \times n$ matrix with $\dfrac{\partial f_i}{\partial y_j}$ as the (i,j) entry; i.e.,

$$
F = \begin{pmatrix}
\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\
\frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\
\vdots & \vdots & & \vdots \\
\frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n}
\end{pmatrix} \tag{3.11}
$$

F is considered as a function of time only, since we have set $\tau=\tau^*, \eta_1=\eta_1^*, \ldots, \eta_n=\eta_n^*$. Notice that <u>F is a numerically known matrix function of time</u> since all formulas for $\frac{\partial f_i}{\partial y_j}$ can be obtained by direct differentiation of the right hand sides of (s) and the arguments $(y_1, \ldots, y_n, t)$ are given by values along the reference; i.e., by the numerically known functions $\varphi_1(t), \ldots, \varphi_n(t)$.

Let Y be the $n \times (n+1)$ matrix

$$
Y = \begin{pmatrix}
\frac{\partial Y_1}{\partial \eta_1} & \frac{\partial Y_1}{\partial \eta_2} & \cdots & \frac{\partial Y_1}{\partial \eta_n} & \frac{\partial Y_1}{\partial \tau} \\
\frac{\partial Y_2}{\partial \eta_1} & \frac{\partial Y_2}{\partial \eta_2} & \cdots & \frac{\partial Y_2}{\partial \eta_n} & \frac{\partial Y_2}{\partial \tau} \\
\vdots & \vdots & & \vdots & \vdots \\
\frac{\partial Y_n}{\partial \eta_1} & \frac{\partial Y_n}{\partial \eta_2} & \cdots & \frac{\partial Y_n}{\partial \eta_n} & \frac{\partial Y_n}{\partial \tau}
\end{pmatrix} \tag{3.12}
$$

Again, each element of Y is considered as a function of time only. By direct comparison, we see that (3.9) and (3.10) are equivalent to the single matrix equation

$$\dot{Y} = F\,Y\,. \tag{3.13}$$

Since F is numerically known, solutions to (3.13) can be generated by direct numerical integration once initial values are known. But the value of Y at $t=\tau^*$ is in fact known in the following way.

By differentiation of (3.3) with respect to η_j, we see that

$$\left(\frac{\partial Y_i}{\partial \eta_j}\right)_{t=\tau} = \frac{\partial (Y_i)_{t=\tau}}{\partial \eta_j} = \delta_{ij}\ , \tag{3.14}$$

the Kronecker delta. By differentiating with respect to τ, we have that at $t=\tau$,

$$\frac{\partial Y_i}{\partial \tau} = -\frac{\partial Y_i}{\partial t}\ . \tag{3.15}$$

But $\dfrac{\partial Y_i}{\partial t} = f_i$ from (s), understanding that the values $t=\tau^*, \eta_1=\eta_1^*, \ldots, \eta_n=\eta_n^*$ are used. Thus at $t=\tau^*$,

$$Y = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & -f_1 \\ 0 & 1 & 0 & \ldots & 0 & -f_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -f_n \end{pmatrix} \tag{3.16}$$

Hence, numerical integration of (3.13) using the initial value at τ^* given by (3.16), will yield, at $t=t_f^*$, numerical values for <u>all</u> needed first partials of the Y_p except the term $\dfrac{\partial Y_p}{\partial t}$. This, however, is simply given (from (s)) by the value of f_p at $y_1=\varphi_1(t_f^*),\ldots,y_n=\varphi_n(t_f^*)$, $t=t_f^*$. This then yields, via. the solution of (3.8), all first partials necessary for the generation of the linear terms of the Taylor's series for the control laws and final time.

We now describe an analogous procedure for generating second order terms. The reader is reminded that the second order partials whose values presently are of interest are all second partials of the Y_p with respect to pairs of arguments in $(\tau,\eta_1,\ldots,\eta_n)$. An equation analogous to (3.9) and (3.10) (or (3.13)) for the second partials can be obtained by differentiation of (3.9) and/or (3.10) prior to substitution of the arguments $\tau=\tau^*,\eta_1=\eta_1{}^*,\ldots,\eta_n=\eta_n{}^*$, followed by these same substitutions. Thus, for example, differentiation of (3.9) with respect to η_ℓ yields

$$\frac{d}{dt}\left(\frac{\partial^2 Y_i}{\partial\eta_\ell\partial\eta_j}\right)=\sum_{u=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 f_i}{\partial y_s\partial y_u}\frac{\partial Y_u}{\partial\eta_j}\frac{\partial Y_s}{\partial\eta_\ell}+\sum_{u=1}^{n}\frac{\partial f_i}{\partial y_u}\frac{\partial^2 Y_u}{\partial\eta_\ell\partial\eta_j}\ . \tag{3.17}$$

By differentiations of (3.9) and (3.10) with respect to τ, similar expressions can be obtained for

$$\frac{d}{dt}\left(\frac{\partial^2 Y_i}{\partial\tau\partial\eta_j}\right)\quad\text{and}\quad\frac{d}{dt}\left(\frac{\partial^2 Y_i}{\partial\tau^2}\right)\ .$$

We treat (3.17) as representative of all three cases.

Assuming that the first order analysis has already been carried out, i.e., assuming that (3.9) and (3.10) (or (3.13)) have been numerically integrated, all expressions on the right side of (3.17) are numerically known functions of time, with the exception of $\frac{\partial^2 Y_u}{\partial\eta_\ell \partial\eta_j}$, which is, after all, the sought-after quantity. For, $\frac{\partial^2 f_i}{\partial y_s y_u}$, considered as a function of the arguments $(y_1, y_2, \ldots, y_n, t)$, can be found by direct differentiation of the right side of (s), and the arguments $y_1, y_2, \ldots, y_n$ are, as in the integration of (3.13), given by $y_1 = \varphi_1(t), \ldots, y_n = \varphi_n(t)$. The terms $\frac{\partial Y_u}{\partial\eta_j}$ $\left(\text{and } \frac{\partial Y_s}{\partial\eta_\ell}\right)$ are numerically known, if the intermediate values of the matrix Y in (3.13) are stored at each step of the numerical integration of (3.13). And finally, $\frac{\partial f_i}{\partial y_u}$ has already been treated in the F matrix of (3.13). It is clear that similar statements apply to the systems for $\frac{d}{dt}\left(\frac{\partial^2 Y_i}{\partial\tau\partial\eta_j}\right)$ and $\frac{d}{dt}\left(\frac{\partial^2 Y_i}{\partial\tau^2}\right)$. Hence, just as in the case of (3.9) and (3.10) (or equivalently, (3.13)), equation (3.17), for each $i, j, \ell = 1, 2, \ldots, n$, can be numerically integrated from $t = \tau^*$ to $t = t_f^*$ to yield numerical values of $\frac{\partial^2 Y_i}{\partial\eta_\ell \partial\eta_j}$ $\left(\text{and } \frac{\partial^2 Y_i}{\partial\tau\partial\eta_j} \text{ and } \frac{\partial^2 Y_i}{\partial\tau^2}\right)$ corresponding to the arguments $t = t_f^*, \tau = \tau^*, \eta_1 = \eta_1^*, \ldots, \eta_n = \eta_n^*$, once initial values (at $t = \tau^*$) are known. These initial values are of three kinds: (i) initial values of partials of the form $\frac{\partial^2 Y_i}{\partial\eta_\ell \eta_j}$, (ii) initial values of partials of the form $\frac{\partial^2 Y_i}{\partial\tau\partial\eta_j}$, and (iii) initial values of partials of the form $\frac{\partial^2 Y_i}{\partial\tau^2}$. We treat them in this order.

Clearly, by (3.14), all partials of the first type, at $t = \tau$, are zero. For partials of the second type, we refer to (3.15). According to (3.15) and (s),

$$\frac{\partial Y_i}{\partial \tau}(\tau,\tau,\eta_1,\ldots,\eta_n) = - f_i(\eta_1,\eta_2,\ldots,\eta_n,\tau) \qquad (3.18)$$

for each $i=1,2,\ldots,n$. In consequence, we have

$$\frac{\partial^2 Y_i}{\partial\tau\partial\eta_j}(\tau,\tau,\eta_1,\ldots,\eta_n) = - \frac{\partial f_i}{\partial y_j}(\eta_1,\eta_2,\ldots,\eta_n,\tau). \qquad (3.19)$$

This equation furnishes the initial values for the second type of partials.

For the third type, we use (3.15) again. It will be clearer, however, to rewrite (3.15) exhibiting all involved arguments.

$$\frac{\partial Y_i}{\partial \tau}(\tau,\tau,\eta_1,\ldots,\eta_n) = - \frac{\partial Y_i}{\partial t}(\tau,\tau,\eta_1,\ldots,\eta_n) \; .$$

Using the chain rule to differentiate once again with respect to τ,

$$\frac{\partial^2 Y_i}{\partial t\partial\tau} + \frac{\partial^2 Y_i}{\partial \tau^2} = - \frac{\partial^2 Y_i}{\partial t^2} - \frac{\partial^2 Y_i}{\partial\tau\partial t} \; ,$$

which gives

$$\frac{\partial^2 Y_i}{\partial \tau^2} = - 2 \frac{\partial^2 Y_i}{\partial\tau\partial t} - \frac{\partial^2 Y_i}{\partial t^2} \; , \qquad (3.20)$$

where the arguments under consideration are $(\tau,\tau,\eta_1,\ldots,\eta_n)$. Now (3.20) will yield the numerical value for $\frac{\partial^2 Y_i}{\partial\tau^2}$ only if values are known for $\frac{\partial^2 Y_i}{\partial\tau\partial t}$ and for $\frac{\partial^2 Y_i}{\partial t^2}$. These are obtained from (3.4):

$$\frac{\partial}{\partial t} Y_i(t,\tau,\eta_1,\ldots,\eta_n)=f_i(Y_1(t,\tau,\eta_1,\ldots,\eta_n),\ldots,$$

$$Y_n(t,\tau,\eta_1,\ldots,\eta_n),t). \tag{3.4}$$

Differentiation with respect to τ yields

$$\frac{\partial^2 Y_i}{\partial\tau\partial t} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}\frac{\partial Y_j}{\partial \tau}, \quad \text{so that at} \quad t = \tau,$$

by (3.18),

$$\left.\frac{\partial^2 Y_i}{\partial\tau\partial t}\right|_{t=\tau} = \sum_{j=1}^{n} - \frac{\partial f_i}{\partial y_j}(\eta_1,\eta_2,\ldots,\eta_n,\tau)f_j(\eta_1,\eta_2,\ldots,\eta_n,\tau). \tag{3.21}$$

The other term in (3.20) is $\dfrac{\partial^2 Y_i}{\partial t^2}$. Differentiation of (3.4) with respect to t yields

$$\frac{\partial^2 Y_i}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}\frac{\partial Y_j}{\partial t} + \frac{\partial f_i}{\partial t}.$$

At $t=\tau$, (since $\dfrac{\partial Y_i}{\partial t} = f_j$),

$$\left.\frac{\partial^2 Y_i}{\partial t^2}\right|_{t=\tau} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}(\eta_1,\ldots,\eta_n,\tau)f_j(\eta_1,\ldots,\eta_n,\tau) +$$

$$+ \frac{\partial f_i}{\partial t}(\eta_1,\ldots,\eta_n,\tau). \tag{3.22}$$

Combining (3.21) and (3.22),

$$\left.\frac{\partial^2 Y_i}{\partial \tau^2}\right|_{t=\tau} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}(\eta_1,\ldots,\eta_n,\tau) f_j(\eta_1,\ldots,\eta_n,\tau) - \frac{\partial f_i}{\partial t}(\eta_1,\ldots,\eta_n,\tau) \tag{3.23}$$

Equation (3.17) and the related equations can now be numerically integrated from $t=\tau^*$ to $t=t_f^*$. As a result, all needed second partials are numerically known with the exception of the partials of the form

$$\frac{\partial^2 Y_i}{\partial t \partial \eta_j}, \frac{\partial^2 Y_i}{\partial t \partial \tau} \text{ and } \frac{\partial^2 Y_i}{\partial t^2} \text{ at } t = t_f^*.$$

However, the first two of these are given by the appropriate members of (3.13) at $t=t_f^*$ while the latter is given by an expression such as (3.22) differing only insofar as the arguments $y_1=\varphi_1(t_f^*),\ldots y_n=\varphi_n(t_f^*)$, $t=t_f^*$ replace the arguments $y_1=\eta_1,\ldots,y_n=\eta_n$, $t=\tau$.

This then yields all necessary information for the determination of the second partials necessary for the generation of the quadratic terms of the Taylor's series for the control laws and final time.

The generalization of the procedure to all orders and types under consideration should at this point be fairly clear. All partials of the type $\dfrac{\partial^p Y_i}{\partial \tau^{p_0} \partial \eta_1^{p_1} \ldots \partial \eta_n^{p_n}}$, $p_0 + p_1 + \ldots + p_n = p$, are obtained by numerical integration of the appropriate analogue of (3.13) or (3.17). Initial values for the integration follow from extensions of the arguments presented for the second order analysis. When dealing with a partial of the type $\dfrac{\partial^{p+q} Y_i}{\partial t^q \partial \tau^{p_0} \partial \eta_1^{p_1} \ldots \partial \eta_n^{p_n}}$, one considers first the right-hand side of the differential equation for $\frac{\partial^p Y_i}{\partial\tau^{p_o}\partial\eta_1^{p_1}\ldots\partial\eta_n^{p_n}}$, i.e., the analogue of (3.13) and (3.17). Symbolically, this may be written

$$\frac{d}{dt}\left(\frac{\partial^p Y_i}{\partial\tau^{p_o}\partial\eta_1^{p_1}\ldots\partial\eta_n^{p_n}}\right) = G_{p_o p_1 \ldots p_n},$$

in which $G_{p_o p_1 \ldots p_n}$ is a function of partials of Y_i of orders less than or equal p_o. Then the expression for

$\frac{\partial^{p+q} Y_i}{\partial t^q \partial\tau^{p_o}\partial\eta_1^{p_1}\ldots\partial\eta_n^{p_n}}$ is obtained by application of the chain rule to effect differentiation of $G_{p_o p_1 \ldots p_n}$ q times with respect to t. Upon completion of this, the arguments $y_1=\varphi_1(t_f^*),\ldots,y_n=\varphi_n(t_f^*)$, $t=t_f^*$ are used to obtain the needed values. As stated previously, we will not be any more explicit than this about treatment of higher order terms since the method is relatively clear and since the notation for the general case is somewhat problematic.

D. EXTENSION TO CERTAIN DISCONTINUOUS REFERENCE TRAJECTORIES: THE PROBLEM OF STAGING

In this, the final section, we consider the problem of defining and generating the control laws and final time neighboring a reference trajectory resulting from the flight of a multistage vehicle. The problem actually treated will be concerned with a two stage configuration; generalization to more stages is immediate. The result of the introduction of a second stage is a slight modification of the definition of the control laws and final time (cf. (2.6)). Following a brief discussion of the nature of the problem of staging, the boundary value problem of Section II is appropriately reformulated.

In formulating the staging problem, we assume first that every stage other than the last stage terminates at a specified time. Therefore, if the vehicle consists of n stages, we assume that there is given a set of n-1 values, $t_{1_f}, t_{2_f}, \ldots, t_{n-1_f}$, which are the times of termination of the first through the n minus first stages, respectively.* In some flights, it is desired to insert coasting periods between stages, or to interrupt one or more stages to insert coasts. For many purposes, a coast can be considered as a stage in which thrust and mass flow rate are zero.

The form of system (s) depends on the vehicle construction, the flight environment, and the optimization (cf. Part D, Section I). For this reasoh, it may well happen that the system (s) will change from one stage to another. While such a situation doesn't pose unsolvable complications, we will nevertheless assume in this treatment that the system (s) is applicable to all stages. There is still sufficient generality to illustrate the application of the methods of the last section to several stages, and application of the methods to more general situations can be viewed as an extension of these procedures.

Recall that in Part A of Section II, following Theorem 1, the initial values (at τ) were divided into two groups, the <u>state parameters</u> and the <u>control parameters</u>. Under the assumption that staging is to occur at prespecified times and other assumptions concerning the vehicle and its environment, it can be shown that the control variables (by which is meant those variables in (s) whose initial values have been designated control parameters) can be taken as continuous across stage junctions. This important result can be found in Reference 3.

Consider now the behavior of the state variables across stage junctions. Some of these are necessarily continuous there; this is clearly true of position and velocity coordinates. However, other state variables can possess determinable jump discontinuities at stage junctions. For example, if separation occurs, there results a discontinuity in mass.

* These times usually correspond essentially to fuel depletion in each of the lower stages.

Generally speaking, discontinuities can be expected in thrust magnitude and propellant flow rate.

At each stage junction, therefore, it is assumed that all control variables and some state variables are continuous, while the remaining state variables possess jump discontinuities. For those variables which are continuous, the initial values after staging are the same as the values at termination of the previous stage. The discontinuous state variables are assumed to have specified initial values in each stage. This is because their values are dependent on the vehicle construction rather than on the flight path.

Bearing all this in mind, we reformulate the boundary value problem of Section II to correspond to a two stage flight of the sort just described. It is assumed, as in that section, that there has been determined numerically a reference trajectory. This reference, however, consists of two stages. Suppose t_{1f} denotes the instant of termination of the first stage of the reference (and, according to prior discussion, the instant of termination of the first stage of every trajectory). We can certainly assume that $\tau < t_{1f}$ for otherwise the problem reduces to that of one stage flight. From the set of state variables, $y_1, y_2, \ldots, y_m$ (corresponding to the state parameters, $\eta_1, \eta_2, \ldots, \eta_m$) suppose $y_1, y_2, \ldots y_p$, $0 \leq p \leq m$ are continuous across stage junctions (and this supposition is made for every trajectory, not just the reference) while $y_{p+1}, y_{p+2}, \ldots y_m$ possess jump discontinuities at the stage junction. Let the initial values of these state variables at the beginning of the second stage be, in every case, $\eta_{p+1}^{(2)}, \eta_{p+2}^{(2)}, \ldots, \eta_m^{(2)}$. Let the functions $\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$, $\tau \leq t \leq t_f^*$, be the numerically known reference values.

The definition of a proper, real, analytic, non-singular, controllable solution is essentially unchanged. The one modification is in the definition of the Jacobian J of Part (v). Just as in Section II, the functions defined in Theorem 1 will have to be used in defining J and, for that matter, the boundary problem itself.

If Theorem 1 is applied to the reference trajectory (assuming the reference satisfies the first four properties of the definition of a P.R.A.N.C. solution) for the t-interval $[\tau, t_{1_f}]$, one obtains the family $Y_i(t,\tau,\eta_1,\ldots,\eta_n)$, $i=1,2,\ldots,n$ on $[\tau, t_{1_f}]$ which reduces to the reference for $\eta_i=\eta_i^*$, $i=1,2,\ldots,n$. Theorem 1 can again be applied to the <u>second stage</u>, yielding the family $Z_i(t,\sigma,\zeta_1,\ldots,\zeta_n)$, $i=1,2,\ldots,n$ on $[t_{1_f}, t_f^*]$, which reduces to the nominal for appropriate initial values. These initial values are $\sigma=t_{1_f}$, and the values of $y_1,\ldots,y_n$ on the reference at the initial point of the second stage of the reference, as opposed to the final point of the first stage. According to the foregoing discussion of discontinuities at t_{1_f}, the following is true. The initial values for $Z_i(t,\sigma,\zeta_1,\ldots,\zeta_n)$ at $\sigma = t_{1_f}$ are related to the terminal values of $Y_i(t,\tau,\eta_1,\ldots,\eta_n)$ at $t=t_{1_f}$ in the following manner:

At $\sigma = t_{1_f}$:

$$\left.\begin{aligned}
\zeta_i &= Y_i(t_{1_f},\tau,\eta_1,\ldots,\eta_n); \quad i=1,2,\ldots,p \\
\zeta_i &= \eta_i^{(2)}; \quad i=p+1,p+2,\ldots,m \\
\zeta_i &= Y_i(t_{1_f},\tau,\eta_1,\ldots,\eta_n); \quad i=m+1,m+2,\ldots,n\,.
\end{aligned}\right\} \qquad (3.24)$$

Equations (3.24) furnish the key to multistage analysis. The first and last lines specify continuity of certain of the states and the controls across the stage junction, while the middle line specifies initial values for the discontinuous states. It is agreed that (3.24) is to hold for <u>all</u> trajectories, and therefore the arguments $\sigma,\zeta_1,\ldots,\zeta_n$ of the Z_i can be replaced by (3.24). In this way, we define functions $Z_i'(t,\tau,\eta_1,\ldots,\eta_n)$ by

$$Z_i'(t,\tau,\eta_1,\ldots,\eta_n)=Z_i(t,t_{1_f},Y_1,\ldots,Y_p,\eta_{p+1}^{(2)},\ldots,\eta_m^{(2)},Y_{m+1},\ldots,Y_n),$$

$$i = 1,\ 2,\ldots,n \qquad (3.25)$$

where, for each Y_i appearing in (3.25), the arguments are $Y_i(t_{1_f},\tau,\eta_1,\ldots,\eta_n)$.

The boundary value problem now becomes the same as before, except that the composite functions $F_j{}^*(t,\tau,\eta_1,\ldots,\eta_m,\eta_{m+1},\ldots,\eta_{m+k})$, $j=1,2,\ldots,k+1$ are given by the composition

$$F_j\Big(t,Z_1'(t,\tau,\eta_1,\ldots,\eta_n),\ldots,Z_n'(t,\tau,\eta_1,\ldots,\eta_n)\Big),$$

and the Jacobian J is correspondingly different.

All techniques of the previous sections are now applicable. One must bear in mind that the $F_j{}^*$ are now doubly composite, since the Z_i' are themselves composite. Thus, for example, in the differentiation of (2.7) to obtain the analogue of (3.5), the chain rule must be applied to the functions Z_i' as well.

Since the system (s) applies to both stages, equations such as (3.13) are still valid for all $t \in [\tau,\ t_f{}^*]$. There is never any need to integrate (3.13) across the discontinuities at t_{1_f}, since the partials needed are given by separate integrations for the Y_i from τ to t_{1_f} and for the Z_i from t_{1_f} to $t_f{}^*$. All of this is plain following differentiation of (2.7), keeping careful track of the involved arguments.

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SPACE VEHICLE GUIDANCE — A BOUNDARY VALUE FORMULATION

By

Robert W. Hunt and Robert Silber

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This document has also been reviewed and approved for technical accuracy.

R. F. Hoelker

R. F. HOELKER, Chief
Astrodynamics & Guidance Theory Div.

E. D. GEISSLER, Director
Aero-Astrodynamics Laboratory

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Mr. J. S. Farrior
Lockheed
P. O. Box 1103
West Station
Huntsville, Alabama

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Farmingdale, Long Island, New York

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Senior Research Engineer
Research Laboratories
United Aircraft Corporation
East Hartford, Connecticut

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Corporate Systems Center
Division United Aircraft Corporation
1690 New Britain Avenue
Farmington, Connecticut

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Sr. Analytical Engineer
Technical Planning
Hamilton Standard Division
United Aircraft Corporation
Windsor Locks, Connecticut

Dr. M. G. Boyce
Department of Mathematics
Vanderbilt University
Nashville, Tennessee

Dr. Daniel Dupree, Project Leader
Department of Mathematics
Northeast Louisiana State College
Monroe, Louisiana

Dr. Steve Hu
Northrop Corporation
Box 1484
Huntsville, Alabama

Dr. William Nesline, Jr.
Analytical Research Department
Missile and Space Division
Raytheon Company
Bedford, Massachusetts

Dr. Robert W. Hunt
Department of Mathematics
Southern Illinois University
Carbondale, Illinois

Mr. Robert Silber
Department of Mathematics
Southern Illinois University
Carbondale, Illinois

Dr. H. Hermes
Mail #A127
Martin Company
P. O. Box 179
Denver 1, Colorado

Mr. R. W. Reck
Martin Company
3313 S. Memorial Parkway
Huntsville, Alabama

Dr. M. L. Anthony
Space Flight Technology
Mail No. A-153
The Martin Company
Denver 1, Colorado

Mr. Samuel Pines
Analytical Mechanics Associates, Inc.
941 Front Street
Uniondale, New York

Dr. W. A. Shaw
Mechanical Engineering Department
Auburn University
Auburn, Alabama

Mr. J. W. Hanson
Computation Center
University of North Carolina
Chapel Hill, North Carolina

Mr. Oliver C. Collins
Advanced Research
Flight Technology Department
Aero-Space Division
Organization 2-5762
Mail Stop 15-12
Boeing Company
P. O. Box 3707
Seattle, Washington

Mr. Carl B. Cox
Chief, Flight Technology Dept.
Advanced Research
Aero-Space Division
Organization 2-5762
Mail Stop 15-03
Boeing Company
P. O. Box 3707
Seattle 24, Washington

Mr. Richard Hardy
Box AB-38
Boeing Company
P. O. Box 1680
Huntsville, Alabama

Mr. Wes Morgan
Box AB-49
Boeing Company
P. O. Box 1680
Huntsville, Alabama

Mr. Robert Glasson
Bendix Systems Division
Bendix Corporation
3322 Memorial Parkway South
Huntsville, Alabama

Chrysler Corporation Missile Division
Sixteen Mile Road and Van Dyke
P. O. Box 2628
Detroit 31, Michigan
ATTN: Mr. T. L. Campbell
Dept. 7162
Applied Mathematics

Mr. Harry Passmore
Hayes International Corporation
P. O. Box 2287
Birmingham, Alabama

Astrodynamics Operation
Space Sciences Laboratory
Missile and Space Vehicle Department
General Electric Company
Valley Forge Space Technology Center
P. O. Box 8555
Philadelphia 1, Pennsylvania
ATTN: Mr. J. P. deVries
Dr. V. Szebehely
Mr. C. Cavoti

Dr. I. E. Perlin
Rich Computer Center
Georgia Institute of Technology
Atlanta, Georgia

Dr. Charles C. Conley
Department of Mathematics
University of Wisconsin
Madison, Wisconsin

Dr. O. R. Ainsworth
Department of Mathematics
University of Alabama
Huntsville, Alabama

Mr. D. Johnson
Astrodynamics Section
Astrosciences Department
Aeronutronic Div. of Ford Motor Co.
Ford Road
Newport Beach, California

Dr. N. N. Puri
Electric Engineering Department
Drexel Institute of Technology
Philadelphia, Pennsylvania

University of Kentucky
College of Arts and Sciences
Department of Mathematics and Astronomy
Lexington, Kentucky
ATTN: Dr. J. C. Eaves

Space Sciences Laboratory
Space and Information Systems Division
North American Aviation, Inc.
Downey, California
ATTN: Dr. D. F. Bender

Mr. H. A. McCarty
Space and Information Systems Division
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California

Mr. J. W. Scheuch
P. O. Box 557
North American Aviation, Inc.
Huntsville, Alabama

Mr. Joe Mason
Space and Information Systems Division
Department 41-595-720
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California

Dr. D. M. Schrello
Director, Flight Sciences
Space and Information Systems Division
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California

Mr. Myron Schall
Supervisor, Powered Trajectory
Space and Information Division
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California

Mr. S. E. Cooper
Space and Information Division
Dept. 41-697-610
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California

Space Sciences Laboratory
Space and Information Systems Division
North American Aviation, Inc.
12214 Lakewood Blvd.
Downey, California
ATTN: Mr. Paul DesJardins
Mr. Harold Bell

Dr. E. R. Rang
Military Products Group
Aeronautical Division
Minneapolis-Honeywell Regulator Co.
Mail Stop #340
Minneapolis, Minnesota

Mr. Dahlard Lukes
M.P.G. - Aeronautical Division
Minneapolis-Honeywell Regulator Co.
2600 Ridgway Road
Minneapolis, Minnesota

Dr. Robert M. L. Baker, Jr.
Astrodynamics Research Center
Lockheed-California Company
Burbank, California

Mr. Ralph Q. Haumacher
A2-863: Space/Guidance & Control
Douglas Aircraft Corporation
3000 Ocean Park Blvd.
Santa Monica, California

Mr. F. A. Hewlett
Manager of Documentation
Federal Systems Division
IBM
6702 Gulf Freeway
Houston 17, Texas

Mr. George Cherry
Massachusetts Institute of Technology
Cambridge, Massachusetts

Mr. William C. Marshall
Reseach Engineer
M. P. G. Research Dept. - Sta #340
Minneapolis-Honeywell Regulator Co.
2600 Ridgway Road
Minneapolis 3, Minnesota

Mr. W. G. Melbourne
Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena 3, California

Dr. Byron D. Tapley
Department of Aerospace Engineering
University of Texas
Austin, Texas

Dr. Hans G. Baussus
Aerospace Group
General Precision
McBride Avenue
Little Falls, New Jersey

Dr. William A. Mersman
Chief,
Electronic Machine Computing Branch
Ames Research Center
Moffett Field, California

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Biotechnology Division
Ames Research Center
Moffett Field, California

Dr. Herman M. Dusek
A. C. Research and Development
A. C. Spark Plug
The Electronics Division
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950 N. Sepulveda Blvd.
El Segundo, California

Mr. Howard S. London
Rm. 910-E
Bellcomm, Inc.
1100 17th Street, N. W.
Washington, D. C.

Mr. Howard Haglund
Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena 3, California

Mr. Hewitt Phillips
Langley Research Center
Hampton, Virginia

Mr. Frank J. Carroll
Systems Requirement Department
Equipment Division
Raytheon Company
40 Second Avenue
Waltham, Massachusetts

Mr. T. Perkins
Chrysler Corporation
HIC Building
Huntsville, Alabama

Mr. Robert Allen
Manager, Huntsville Sales Office
A. C. Spark Plug
The Electronics Division
of General Motors
Holiday Office Center
Huntsville, Alabama

Mr. Dale B. Ruhmel
Staff Associate
Aeronautical Research Associates of Princeton, Inc.
50 Washington Road
Princeton, New Jersey

Dr. Raymond Rishel
Mathematical Analysis Staff
Organization 2-5330
Mail Stop 89-75
Boeing Company
P. O. Box 3707
Seattle, Washington

Dr. Rudolf Hermann
Director
University of Alabama Research Institute
4701 University Avenue, N.W.
Huntsville, Alabama

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Physical Sciences Laboratory
Army Missile Command, Bldg. 5429
Redstone Arsenal, Alabama

Mr. R. J. Hayes
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Mr. Harold Chestnut
1 River Road
Schenectady, New York

Space Sciences Laboratory
Space and Information Systems
North American Aviation, Inc.
Downey, California
ATTN: Mr. Dave Engles

Dr. Kirk Brouwer
Yale University Observatory
Box 2023, Yale Station
New Haven, Connecticut

Dr. Imre Izsak
Smithsonian Institution Astrophysical Observatory
60 Garden Street
Cambridge 38, Massachusetts

Dr. Peter Musen
Goddard Space Flight Center
NASA
Greenbelt, Maryland

Dr. Yoshihide Kozai
Smithsonian Institution Astrophysical Observatory
60 Garden Street
Cambridge 38, Massachusetts

Dr. Rudolph Kalman
Research Institute for Advanced Study
7212 Bellona Avenue
Baltimore 12, Maryland

Mr. Ken Kissel
Aeronautical Systems Division
Applied Mathematics Research Branch
Wright-Patterson Air Force Base
Dayton, Ohio

Mr. Jack Funk
Manned Spacecraft Center
Flight Dynamics Branch
NASA
Houston, Texas

Dr. J. B. Rosser
Department of Mathematics
Cornell University
Ithaca, New York

Douglas Aircraft Corporation
3000 Ocean Park Blvd.
Santa Monica, California
ATTN: R. E. Holmen A2-263
Guidance & Control Section

Dr. Joseph F. Shea
Office of Manned Space Flight
National Aeronautics and Space Administration
801 19th Street, N. W.
Washington 25, D. C.

Mr. Charles F. Pontious
Guidance & Navigation Program
Office of Advanced Research & Technology
Code: REG
National Aeronautics and Space Administration
Washington 25, D. C.

Mr. Jules Kanter
Guidance & Navigation Program
Office of Advanced Research & Technology
Code: REG
National Aeronautics and Space Administration
Washington 25, D. C.

Dr. Ray Wilson
OART-Code RRA
Washington 25, D. C.

Dr. Joseph W. Siry
Theory & Analysis Office (547)
Data Systems Division
Goddard Space Flight Center
Greenbelt, Maryland

Douglas Aircraft Corporation
3000 Ocean Park Blvd.
Santa Monica, California
ATTN: Mr. Joe Santa
A2-863

J. B. Cruz, Jr.
Research Associate Professor
Coordinated Science Laboratory
Urbana, Illinois

Mr. Alan L. Friedlander
Research Engineer
Guidance and Control Section
IIT Research Center
10 W. 35th Street
Chicago 16, Illinois

Mr. Joseph V. Natrella
Manned Space Flight - Code MCR
NASA Headquarters, FOB 10B
Washington, D. C.

Mr. Donald Jezewski
Guidance Analysis Branch
Spacecraft Technology Division
Manned Spacecraft Center
Houston, Texas

Mr. Robert M. Williams
Chief, Guidance Analysis
General Dynamics/Astronautcs
Mail Zone 513-0
P. O. Box 166
San Diego 12, California

Mr. M. D. Anderson, Jr.
General Dynamics Corporation
Suite 42 Holiday Office Center
South Memorial Parkway
Huntsville, Alabama

Mr. Y. L. Luke
Mathematics & Physics Division
Midwest Research Institute
425 Volker Boulevard
Kansas City 10, Missouri

Mr. Ted Guinn
Advanced Space Technology
Engineering Research
Douglas Aircraft Corporation
Santa Monica, California

Mr. Robert Chilton
NASA Manned Spacecraft Center - EG
P. O. Box 1537
Houston, Texas

Dr. S. E. Ross
Office of Manned Space Flight - NASA
Washington, D. C.

Stephen J. Kahne
Lt. USAF
Applied Mathematics Branch
Data Sciences Laboratory
Air Force Cambridge Research Laboratories
Office of Aerospace Research
Lawrence G. Hanscom Field
Bedford, Massachusetts

Mr. Daniel B. Killeen
Computer Laboratory
Norman Mayer Bldg.
Tulane University
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Dr. Bernard Friedland
Staff Scientist - Control
General Precision, Inc.
Little Falls, New Jersey

Mr. Walter L. Portugal
Manager, Systems Sales
Aerospace Group
General Precision, Inc.
Little Falls, New Jersey

Mr. C. H. Tross
Manager, Aerospace Sciences
UNIVAC Division of Sperry Rand Corporation
P. O. Box 6248
San Diego 6, California

Mr. Ken Squires
Goddard Space Flight Center, Bldg. #1
National Aeronautics and Space Administration
Greenbelt, Maryland

Dr. Paul Degrarabedian
Astro Science Laboratory
Building G
Space Technology Laboratory, Inc.
One Space Park
Redondo Beach, California

Dr. George Leitmann
Associate Professor, Engineering Science
University of California
Berkeley, California

Dr. R. P. Agnew
Department of Mathematics
Cornell University
Ithaca, New York

Dr. Jurgen Moser
Professor of Mathematics
Graduate School of Arts and Science
New York University
New York, New York

Dr. Lu Ting
Department of Applied Mechanics
Polytechnic Institute of Brooklyn
333 Jay Street
Brooklyn 1, New York

Mr. Clint Pine
Department of Mathematics
Northwestern State College
Natchitoches, Louisiana

Dr. John Gates
Jet Propulsion Laboratory
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Theoretical Guidance & Control Branch
NASA-Ames Research Center
Moffett Field, California

Lt. Col. R. A. Newman
Air Force Space Systems Div
M-SSVH
Bldg. 5250
Redstone Arsenal, Alabama